

## **M<sub>4</sub>: Complex phase, QFT, and QPE**

Complex amplitudes and phase gates extend the real-amplitude models. QFT and QPE need complete worked derivations.

## From signs to complex phases

Earlier models used real amplitudes and relative signs.  $M_4$  allows continuously varying complex phases.

Keep the distinction between a shared global phase and relative phases that affect interference.

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# STEERING OUTCOME PROBABILITIES

**A quantum state:**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

**Measurement outcomes:**

- $|\alpha|^2$ : probability of measuring 0
- $|\beta|^2$ : probability of measuring 1

**Applying a gate changes:**

amplitudes  $\alpha$  and  $\beta$  (coefficients of  $|0\rangle$  and  $|1\rangle$ )

the relative phase between  $|0\rangle$  and  $|1\rangle$

(the relative phase controls interference between paths through the system)

Recall from lecture 9 how quantum systems interfere...

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# USING QUANTUM RULES TO FIND THE PROBABILITY OF LANDING AT POSITION X

Expanding the Quantum Rule:  $|\alpha_1 + \alpha_2|^2$

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Step 1: Magnitude definition:

|ψ

Step 2: Distribute conjugate  
 $= (\alpha_1 + \alpha_2)(\overline{\alpha_1} + \overline{\alpha_2})$

Step 3: Multiply out

$$= \alpha_1 \overline{\alpha_1} + \alpha_1 \overline{\alpha_2} + \alpha_2 \overline{\alpha_1} + \alpha_2 \overline{\alpha_2}$$

Step 4: Recognise terms

$$= |\alpha_1|^2 + |\alpha_2|^2 + \alpha_1 \overline{\alpha_2} + \alpha_2 \overline{\alpha_1}$$

App

Step 5: Combine conjugate pair

$$\alpha_1 \overline{\alpha_2} + \alpha_2 \overline{\alpha_1} = 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

ampl

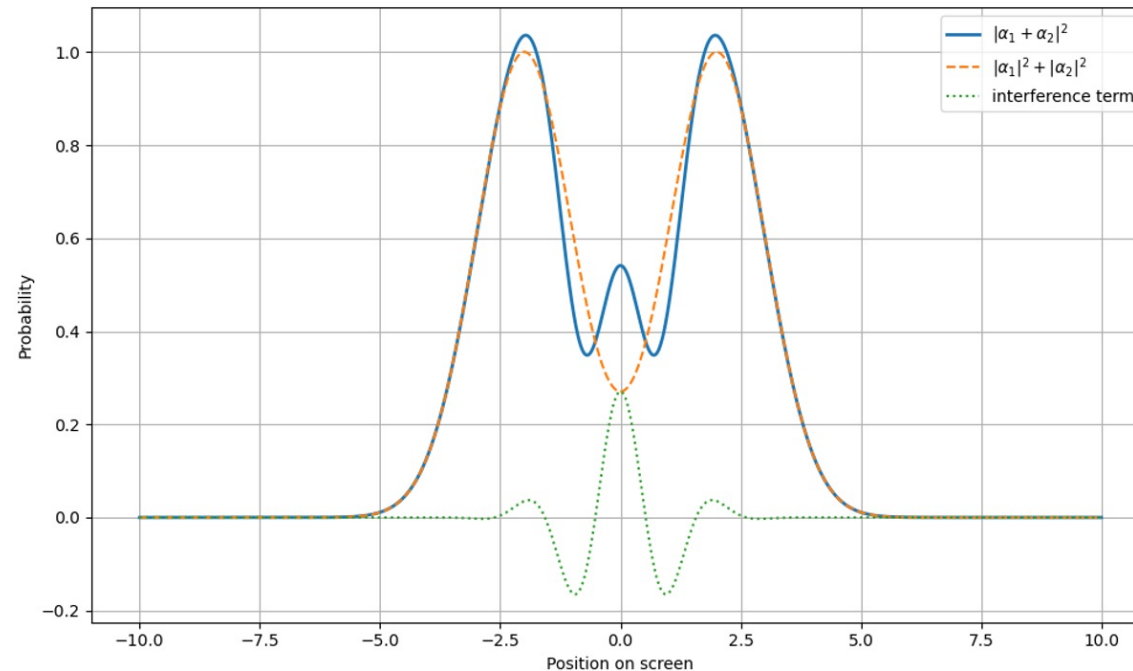
Finally:

$$|\alpha_1 + \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

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$\text{Re}(z)$  means: the real part of a complex number  $z$



Recall from lecture 9 how quantum systems interfere...

## USING QUANTUM RULES TO FIND THE PROBABILITY OF LANDING AT POSITION X

We are plotting continuous Gaussian functions under the *position basis* to produce nice plots to show interference.

In Quantum Computing we are in the discrete *computation basis*, but its harder to plot interference then. It is still happening though.

Recall from lecture 9 how quantum systems interfere...

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# INTERFERENCE AS A COMPUTATIONAL TOOL

From Lecture 9:

$$|\alpha_1 + \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

A quantum state in the **position basis**:

$$|\psi\rangle = \int \psi(x), |x\rangle, dx$$

with  $\psi(x) = \phi_1(x) + \phi_2(x)$

**We can play with the wavefunctions:  $\phi_1(x), \phi_2(x)$**

Separation: shifts where contributions are located

width  $\sigma$ : controls how spread out they are

wave  $k$ : controls oscillation (phase variation across space)

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We are going to look at the quantum state in two basis: the discrete **computational basis** and the continuous **position basis**. The position basis helps me make 2D plots.

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**We can play with the wavefunctions:  $\phi_1(x), \phi_2(x)$**

Separation: shifts where contributions are located

$$\phi(x) \rightarrow \phi(x - d)$$

corresponds to:

$$U(d) = e^{-ipd}$$

p = momentum operator

this is a **translation operator**

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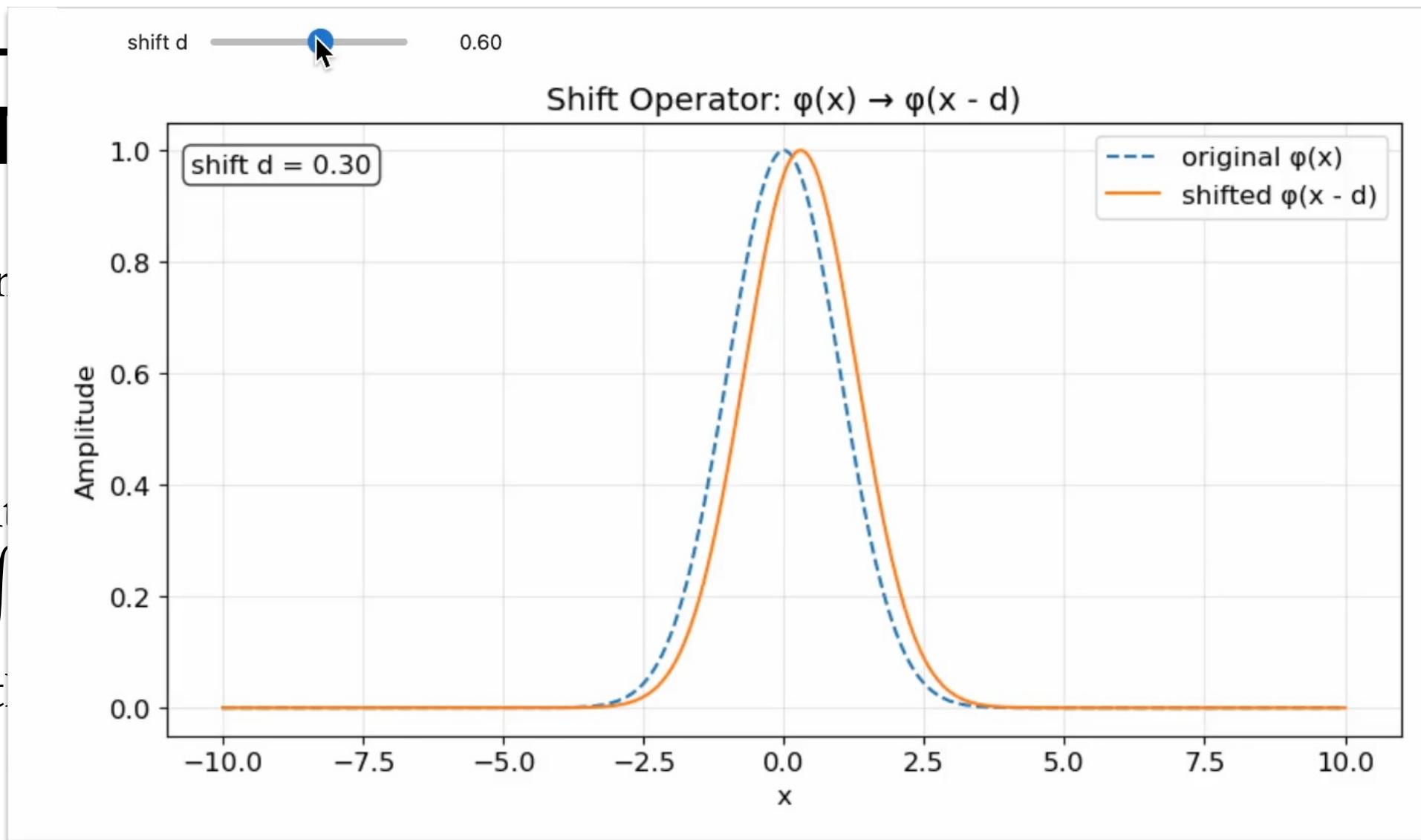
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From

A quantum

$|\psi\rangle = \int$

with



$\phi_1(x)$ ,  $\phi_2(x)$   
created

this is a translation operator

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# INTERFERENCE AS A COMPUTATIONAL TOOL

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A quantum state in the **position basis**:

$$|\psi\rangle = \int \psi(x) |x\rangle dx$$

with  $\psi(x) = \phi_1(x) + \phi_2(x)$

**We can play with the wavefunctions:  $\phi_1(x)$ ,  $\phi_2(x)$**

Position frequency (K) transformation:

$$\phi(x) \rightarrow e^{ikx} \phi(x)$$

k = spatial frequency (how fast phase changes with position)

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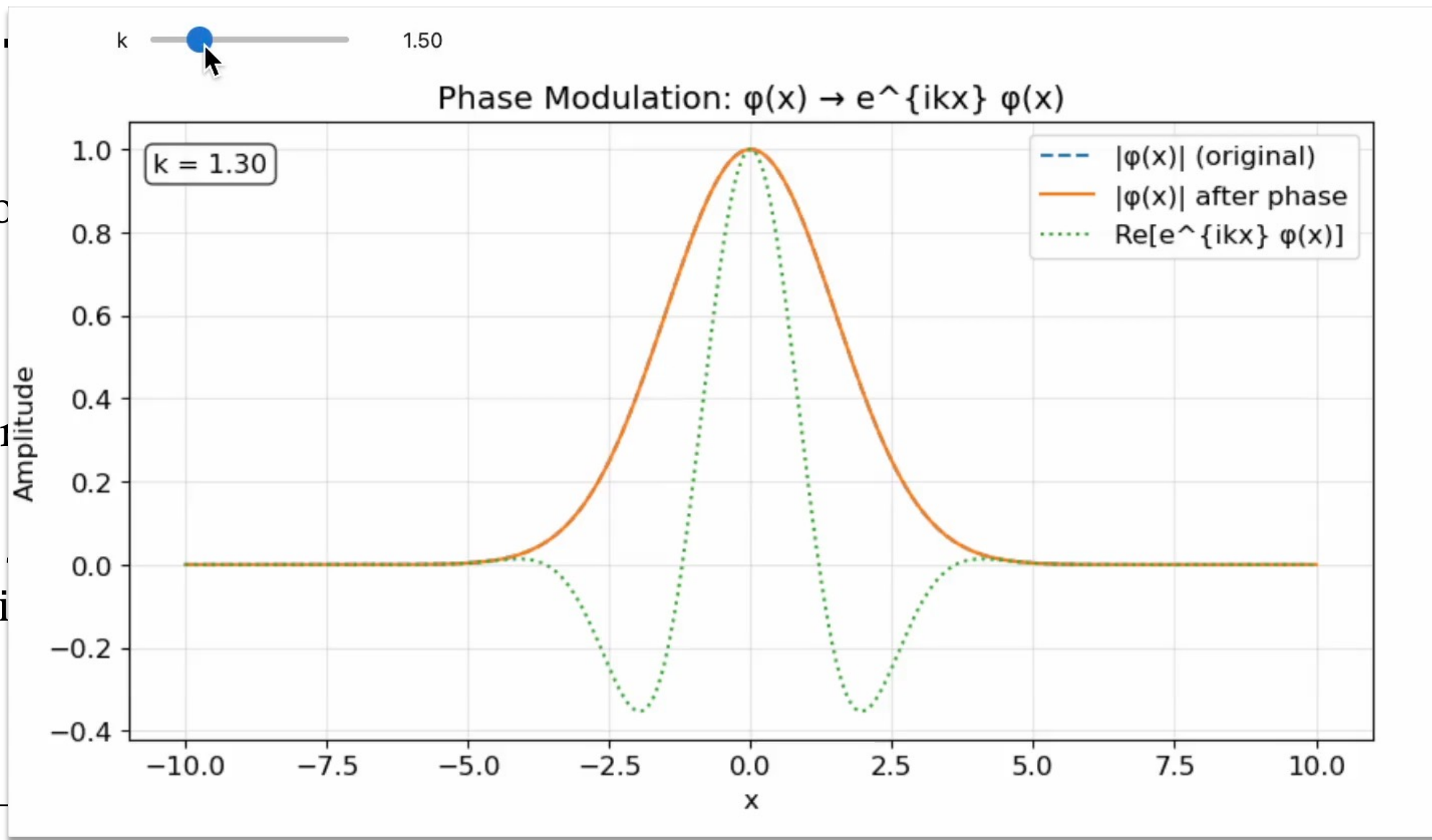
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$|\psi\rangle =$

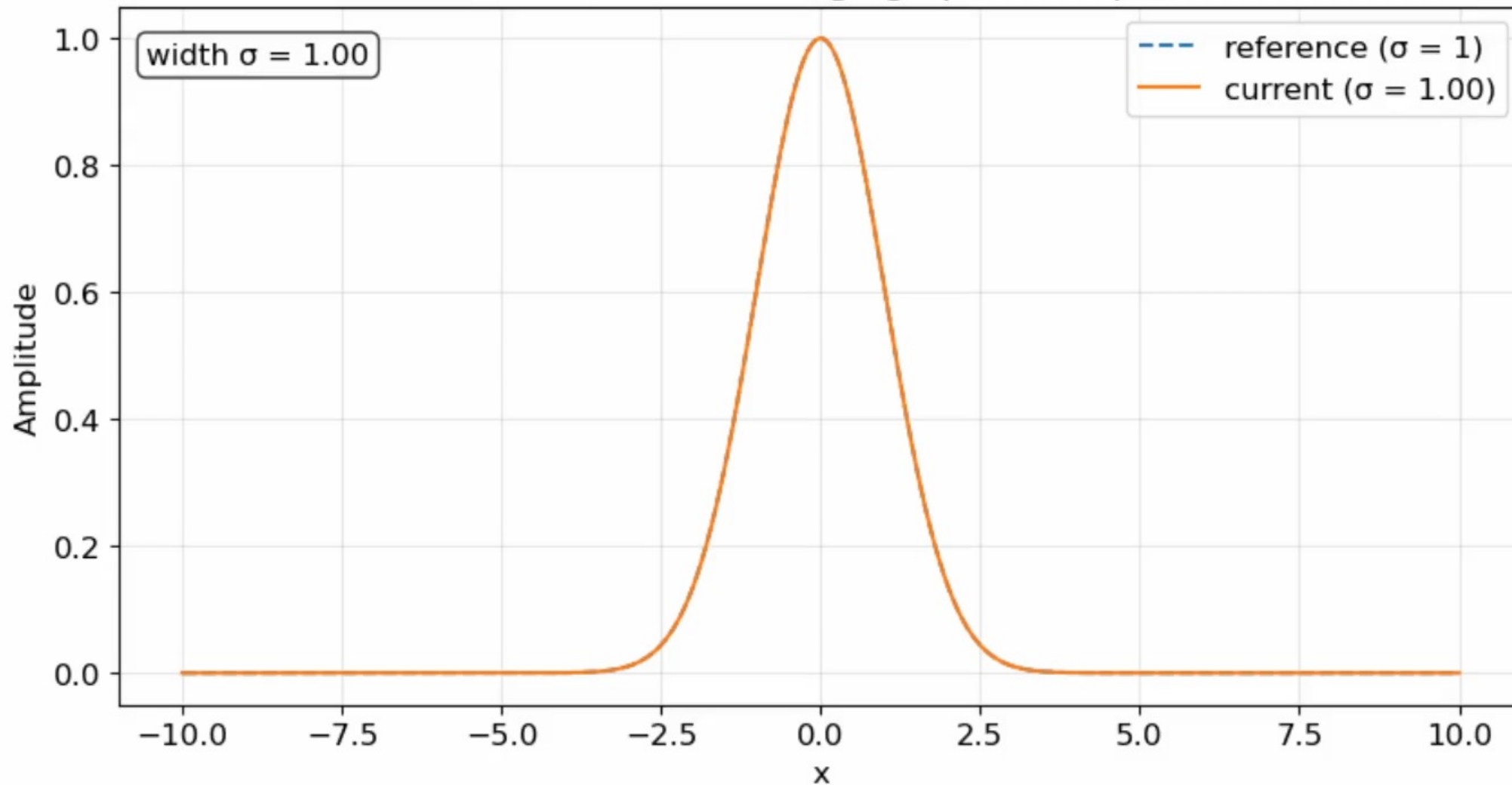
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$\phi_2(x)$   
ted

$\sigma$  1.00

### Width Control: Changing Spread of $\phi(x)$



$A$   
 $|\psi$

$\phi(x)$

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# INTERFERENCE AS A COMPUTATIONAL TOOL

From Lecture 9:

$$|\alpha_1 + \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

A quantum state in the **position basis**:

$$|\psi\rangle = \int \psi(x) |x\rangle dx$$

with  $\psi(x) = \phi_1(x) + \phi_2(x)$

Phase change

The transformation:

$$\psi(x) \rightarrow e^{i\phi} \psi(x)$$

is produced by the operator:

$$U(\phi) = e^{i\phi} I$$

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phase  $\phi$  0.00

Global Phase:  $\psi(x) \rightarrow e^{i\phi}\psi(x)$

phase  $\phi = -0.00$  rad

---  $|\psi(x)|$   
—  $\text{Re}[e^{i\phi}\psi(x)]$   
—  $\text{Im}[e^{i\phi}\psi(x)]$

Amplitude

$|\psi\rangle$

0.0

0.2

0.4

0.6

0.8

1.0

-10.0

-7.5

-5.0

-2.5

0.0

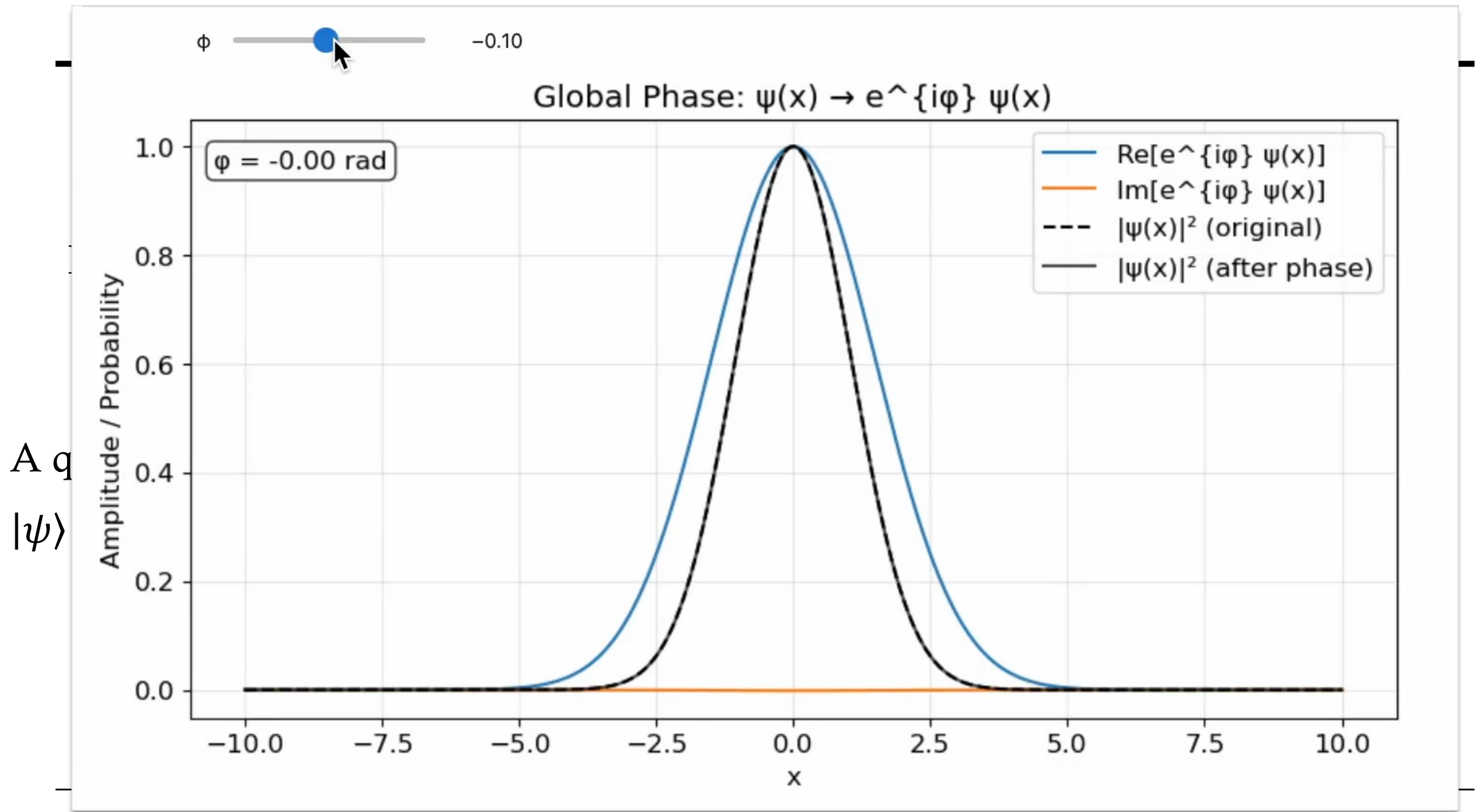
2.5

5.0

7.5

10.0

x



Changing phase doesn't change the probability distribution of the measurement.

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# INTERFERENCE AS A COMPUTATIONAL TOOL

From Lecture 9:

$$|\alpha_1 + \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

Amplitude Change:

The transformation is a complex rotation:

A quantum state in the **computational basis**:

$$U = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\alpha' = \alpha \cos \theta - \beta \sin \theta$$

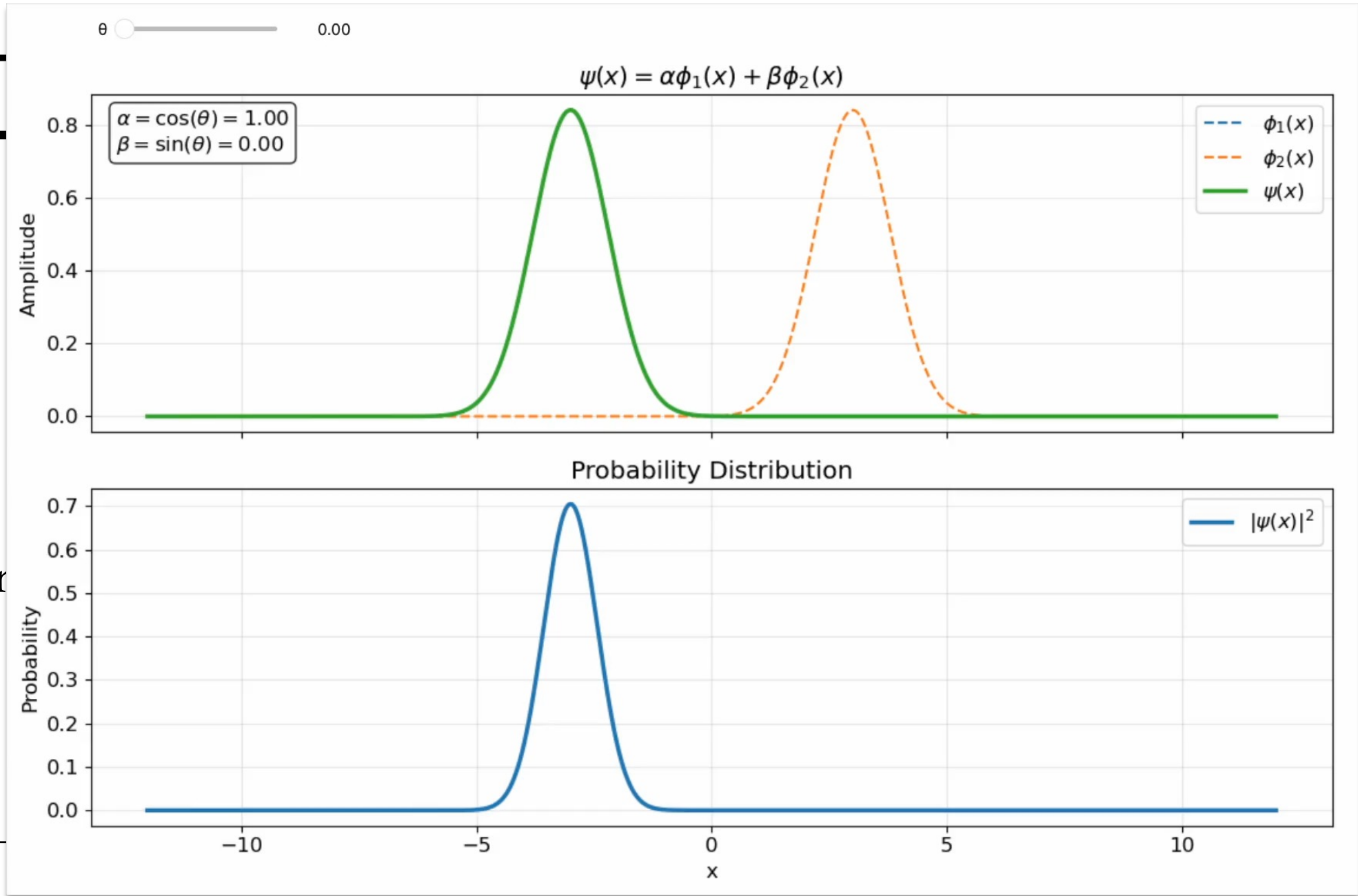
$$\beta' = \alpha \sin \theta + \beta \cos \theta$$

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# INTERFERENCE AS A COMPUTATIONAL TOOL

From Lecture 9:

$$|\alpha_1 + \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 + 2\text{Re}(\alpha_1 \overline{\alpha_2})$$

A quantum state in the **position basis**:

$$|\psi\rangle = \int \psi(x) |x\rangle dx$$

with  $\psi(x) = \phi_1(x) + \phi_2(x)$

Recall that the phase doesn't change the measurement probability distribution.

But it changes the amplitude interactions which does change measurement distributions.

If I multiply the whole state by one phase, nothing measurable changes. But if I change the phase of one component relative to another, I change whether the amplitudes reinforce or cancel, and that changes what I measure.

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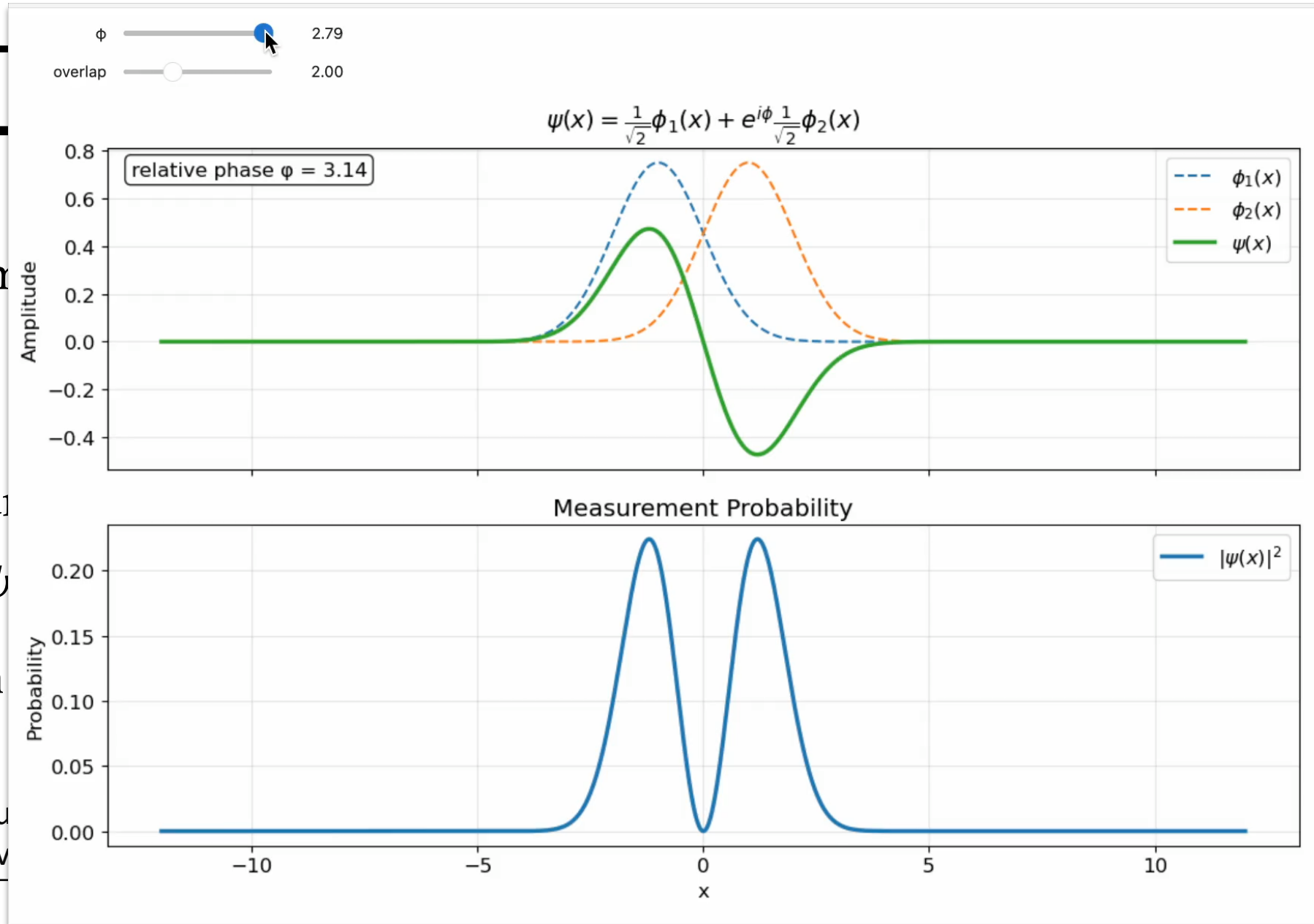
From

A quantum

$$|\psi\rangle = \int \psi$$

with

If I mu  
relativ



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e measurement

is which does

of one component  
measure.

From Lecture 9:

$$\begin{aligned} |\alpha_1 + \alpha_2|^2 &= |\alpha_1|^2 + |\alpha_2|^2 \\ &+ 2\text{Re}(\alpha_1 \overline{\alpha_2}) \end{aligned}$$

**phase  $\Delta$**  → Controls reinforcement or cancellation

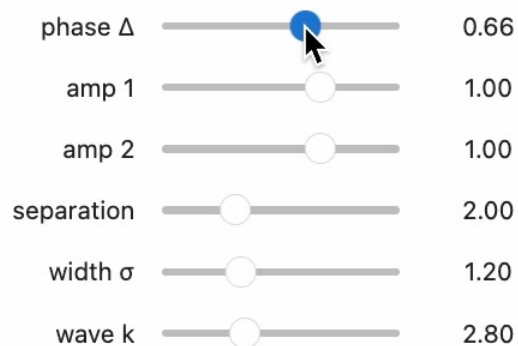
**amp 1** → Strength of path one

**amp 2** → Strength of path two

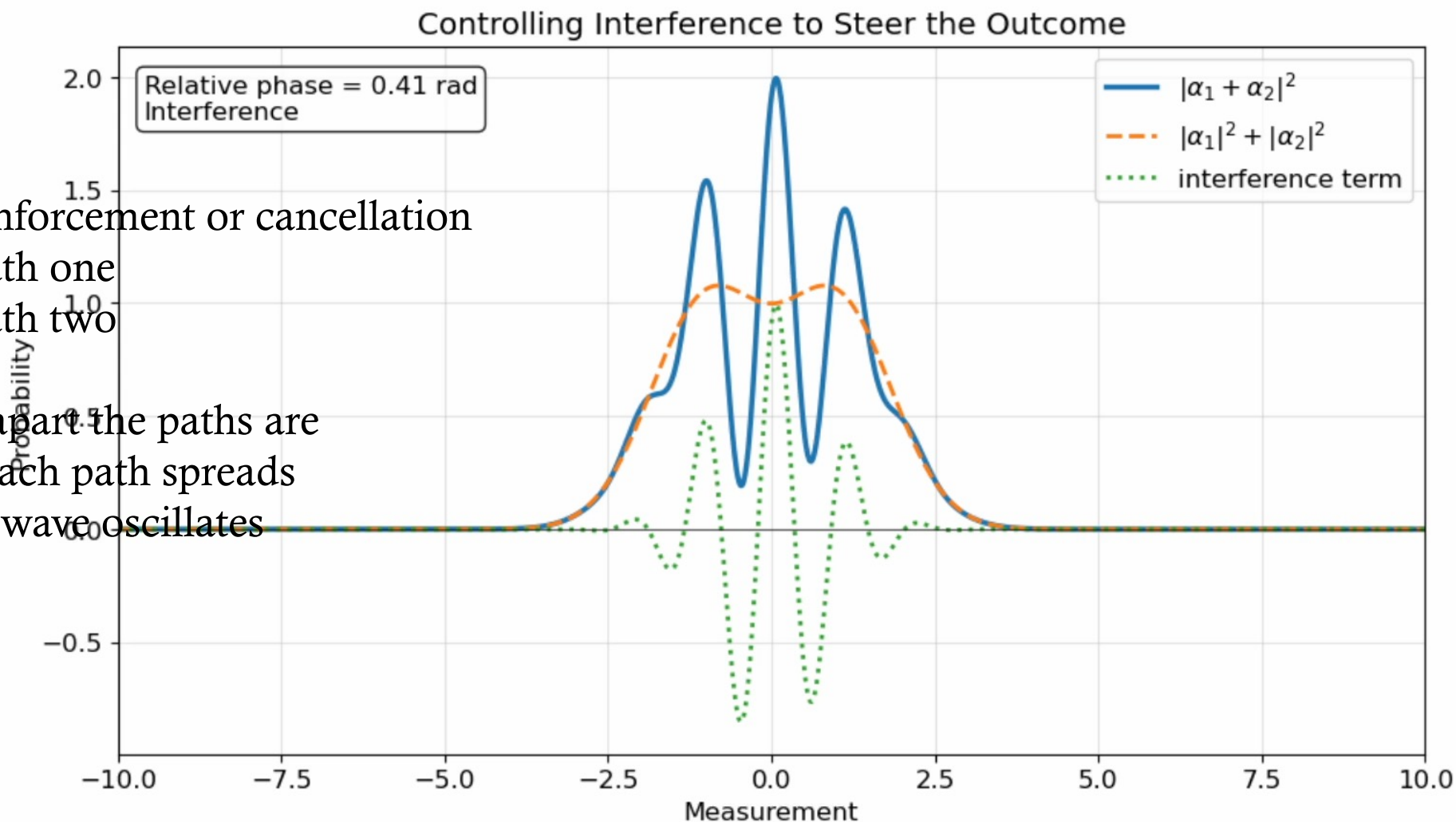
**separation** → How far apart the paths are

**width  $\sigma$**  → How wide each path spreads

**wave  $k$**  → How fast the wave oscillates



# INTERFERENCE AS A COMPUTATIONAL TOOL



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# THE CENTRAL QUESTION

How does a circuit convert phase periodicity into computational-basis structure?

Controlled phase rotations and Hadamards implement the discrete Fourier transform on amplitudes.

Prediction: state what should change in the first experiment and what must remain invariant.

## KEY TERMS

phase gradient  
controlled rotations  
bit reversal  
Fourier peaks

## MODEL

$|x\rangle$  maps to  $\frac{1}{\sqrt{N}} \sum_k \exp(2 \pi i xk/N) |k\rangle$

# Quantum Phase Estimation (QPE)

Goal: Estimate the phase  $\phi$  in an eigenvalue equation:

$$U|\psi\rangle = e^{2\pi i\phi}|\psi\rangle$$

Why it matters:

QPE is the quantum routine that allows us to find periodicity (used in Shor's Algorithm), eigenvalues (in quantum chemistry), and simulate quantum systems.

QPE uses **QFT<sup>-1</sup>** to decode a phase from a quantum state.

- The **phase is stored as a pattern of amplitudes**, and the inverse QFT translates this interference pattern into a readable binary number.

$f(x)$

# Phase gates and roots of unity

Define the phase gate and controlled phase operation.

Use roots of unity to work through a small Fourier transform, including its normalization and basis-order convention.

# QFT and quantum phase estimation

Derive a small QFT and its inverse. Then specify an eigenstate input, controlled powers of  $U$ , and the QPE readout.

Trace an exactly representable phase and explain finite-precision outcomes.