

Tensor products and system composition

States of composite systems. Prerequisites: vector spaces, inner products, and matrix operators.

CLASSICAL SYSTEMS

If two classical systems have states

A has states:

$\{a_1, a_2\}$

B has states:

$\{b_1, b_2\}$

Then the combined system has states

$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)$

This is a **Cartesian product**.

Number of states multiplies:

$$|A \times B| = |A| \cdot |B|$$

CLASSICAL COMPUTING

A classical bit has two states:

0

1

Two bits produce four states:

00

01

10

11

Each state is **distinct and independent**.



QUANTUM COMPUTING

Quantum States

In quantum computing the state of the computer is represented with qubits. A qubit is **not** a discrete value.

It is a vector:

$$|\psi\rangle = (\alpha\beta)$$

with

$$|\alpha|^2 + |\beta|^2 = 1$$

So a qubit lives in $\mathbb{C}^2$.

That means the state space is **continuous**.

Recall in lecture 5 we saw that normalized vectors in $\mathbb{C}^2$ lie on a **3-sphere**.

THE PROBLEM OF COMPOSITION

Suppose we have two qubits.
Each one lives in $\mathbb{C}^2$

How should we represent the **joint system**?

Possible options:

1. Addition
2. Cartesian product
3. Something else

Quantum mechanics chooses something else.

Dimensions and physical states

Two two-level systems have four complex amplitudes. Normalization removes one real degree of freedom, and global phase represents the same physical pure state.

Exponential state-vector size alone does not establish a computational speedup.

TENSOR PRODUCT OF VECTORS

If $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then

$$a \otimes b = \begin{pmatrix} b_1 a_1 \\ b_2 a_1 \\ b_1 a_2 \\ b_2 a_2 \end{pmatrix}$$

BASIS STATES AND DIRAC NOTATION

Basis States of Two Qubits

At this point in the course you are about to start writing objects like

A single qubit lives in the vector space

$$e_0 \otimes e_0, \quad e_0 \otimes e_1, \quad e_1 \otimes e_0, \quad e_1 \otimes e_1$$

$\mathbb{C}^2$
with the standard basis vectors

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Very quickly this becomes visually messy.

Dirac notation compresses this into

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

which is **exactly why physicists invented it**.

BASIS STATES AND DIRAC NOTATION

Two-qubit basis

Tensor products become

$$|0\rangle \otimes |0\rangle = |00\rangle$$

$$|0\rangle \otimes |1\rangle = |01\rangle$$

$$|1\rangle \otimes |0\rangle = |10\rangle$$

$$|1\rangle \otimes |1\rangle = |11\rangle$$

These four vectors form a basis of $\mathbb{C}^4$.

A classical computer with two bits can be in **one of four states**.

The state of a quantum computer with two qubits is **any vector in a four dimensional complex space**.

i.e. an infinite state space.



CLASSICAL VS QUANTUM COMPUTER STATES

Classical Computer
(2 bits)

00

01

10

11

11

Set of 4
states

State = one element
of a finite set

Quantum Computer
(2 qubits)

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$$

$|00\rangle$

$|01\rangle$

$|10\rangle$

$|10\rangle$

$|11\rangle$

State = a vector in a four-
dimensional complex space

AMPLITUDES

Historically, quantum mechanics was first developed as a **wave theory**.

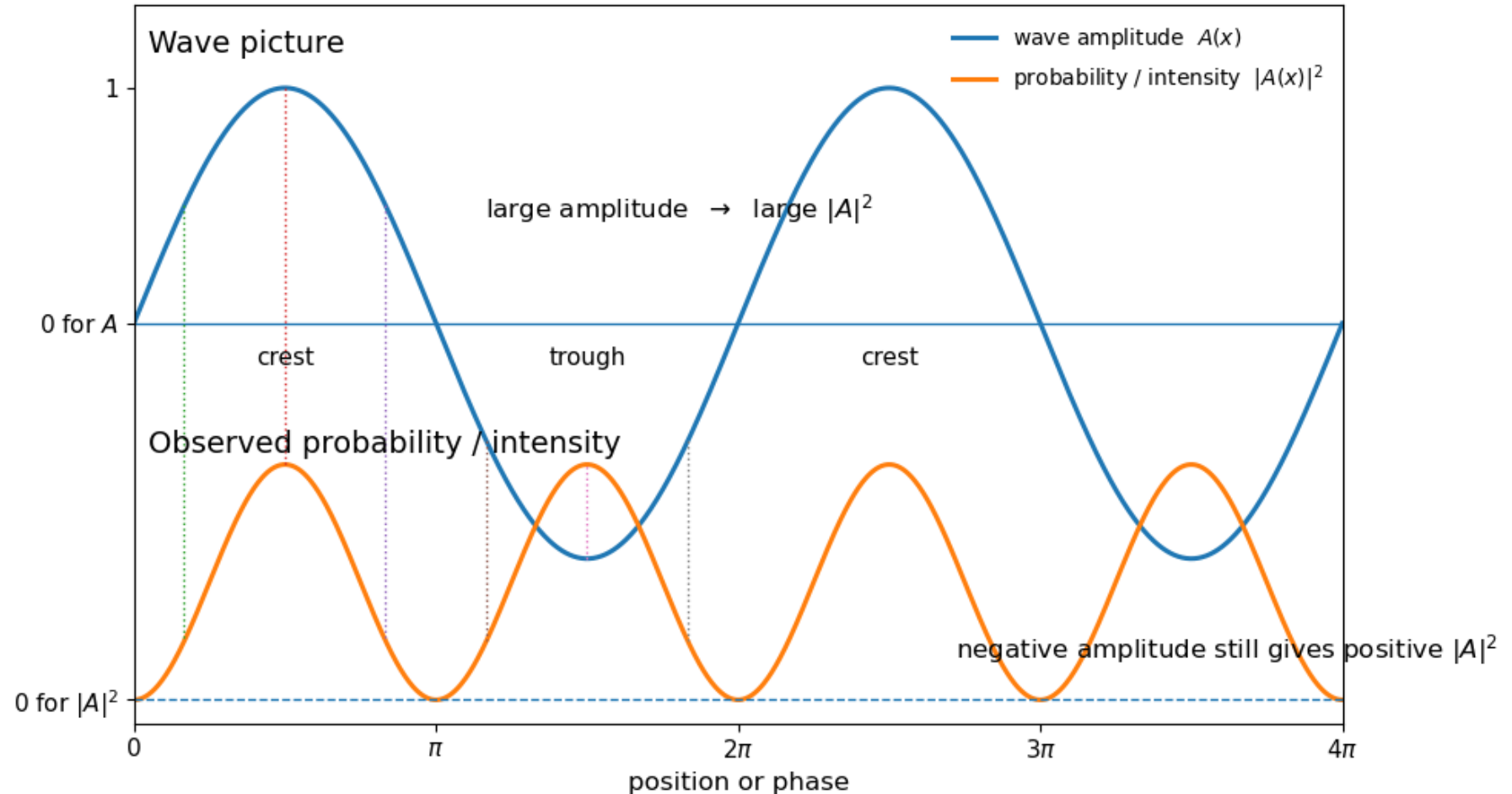
The mathematical structure looked like wave equations, so physicists reused the terminology.

For waves:
intensity $\propto (\text{amplitude})^2$

Quantum mechanics uses the same rule:

$$\text{probability} = |\text{amplitude}|^2$$

From wave amplitude to probability



COMPLEX NUMBERS CAPTURE WAVES

A wave has two important properties:

1. Magnitude (how big it is)

2. Phase (where it is in its cycle)

For example, a sinusoidal wave

$$A \cos(\omega t + \phi)$$

has

- amplitude A
- phase ϕ

Complex numbers encode both of these **in a single object**.

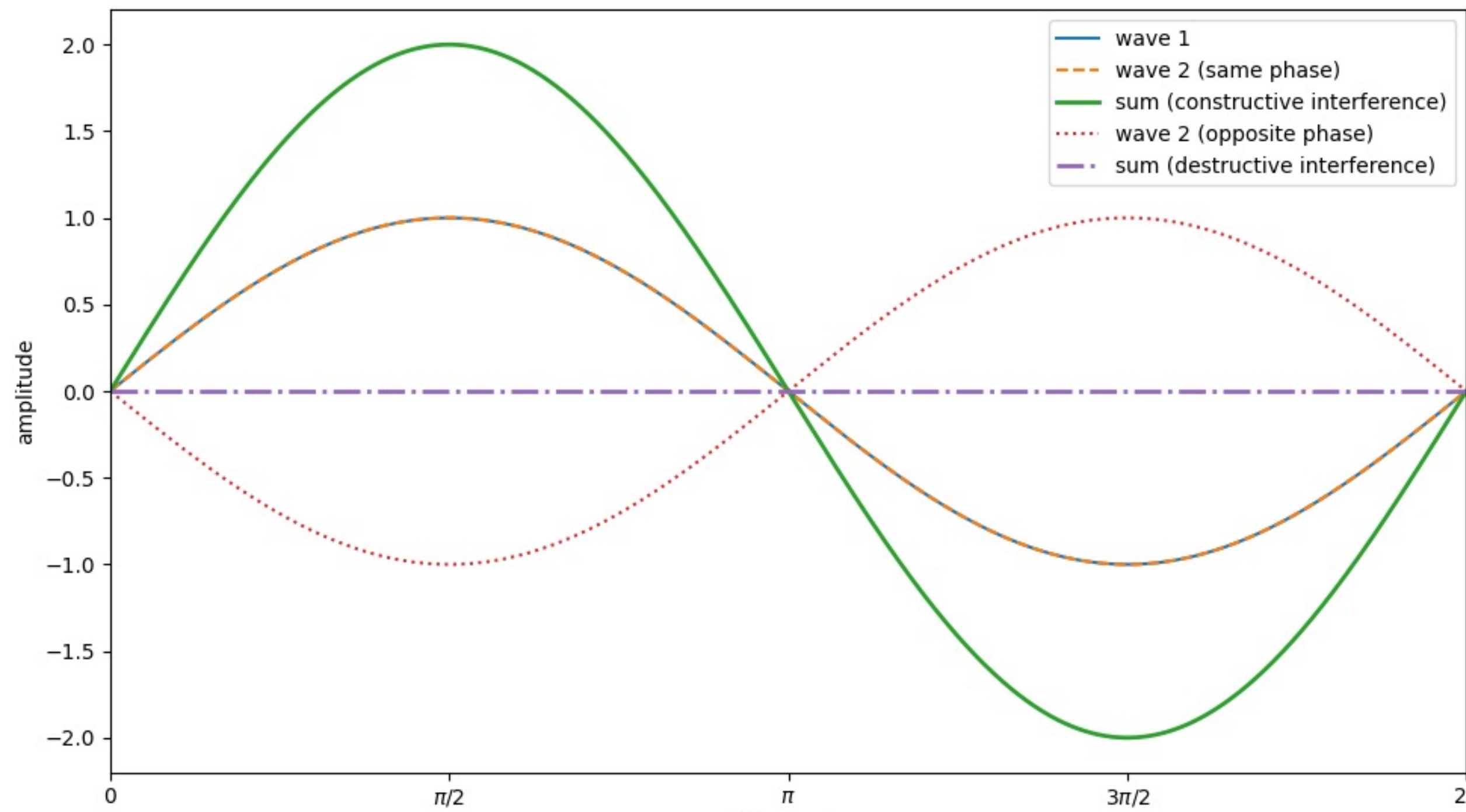
$$Ae^{i\phi}$$

Here

- A is the magnitude
- ϕ is the phase (the angle)

So a complex number is basically a **compact representation of a rotating wave**.

Constructive vs Destructive Interference of Waves



FROM AMPLITUDES TO PROBABILITIES IN MEASUREMENT (THE BORN RULE)

Quantum state (two qubits)

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

The coefficients α_{ij} are **complex amplitudes**.

Born proved that the probability of each outcome is

$$P(ij) = |\alpha_{ij}|^2$$

All the normalization to 1 that creates the spheres is because the probabilities of the states have to sum to 1.



Max Born

“Zur Quantenmechanik der Stoßvorgänge”
(*On the Quantum Mechanics of Collision Processes*), 1926
University of Göttingen

“The probability of finding the particle at a given point is proportional to the square of the wave function.”

*Born was describing the probability of observing a particle at a particular position.

MEASUREMENT – QUANTUM BECOMES CLASSICAL

A quantum computer ultimately produces **classical bits** when we read it.

For two qubits the possible outcomes are

00, 01, 10, 11

These correspond to the basis states

$|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$

Measurement returns **one of these basis states**.

A quantum state determines the probability of each outcome.

QUANTUM SYSTEMS

A **system** is the physical object whose quantum state we are describing.

Examples:

- an electron spin
- a photon polarization
- a trapped ion
- a superconducting circuit

In quantum computing we call such a two state system a **qubit**.

QUANTUM SYSTEMS

Often we study **multiple systems together**.

Example:

Two qubits.

We treat them as a **combined system**.

The possible basis states are

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

This larger system has a **four dimensional state space**.

QUANTUM SYSTEMS

Each individual qubit inside the combined system is called a **subsystem**.

Example:

For the state

$$|10\rangle$$

first qubit = 1

second qubit = 0

So the two qubits are **subsystems of the larger system**.

Product states and nonseparable states

Add a worked tensor product with the basis order stated.

Contrast it with a normalized two-system state that cannot factor into a product. Use state vectors without circuit programming.

Physical meaning of a subsystem

Choose a physical two-level example. Identify the preparation and the measured observable for each subsystem.

Explain which assumptions let us model the pair as a tensor product.