

M₃: Composite quantum systems

Tensor-product review, controlled operations, Bell-state preparation, and entanglement.

TENSOR PRODUCT OF VECTORS

If $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then

$$a \otimes b = \begin{pmatrix} b_1 a_1 \\ b_2 a_1 \\ b_1 a_2 \\ b_2 a_2 \end{pmatrix}$$

BASIS STATES AND DIRAC NOTATION

Basis States of Two Qubits

A single qubit lives in the vector space

$\mathbb{C}^2$
with the standard basis vectors

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

At this point in the course you are about to start writing objects like

$$e_0 \otimes e_0, \quad e_0 \otimes e_1, \quad e_1 \otimes e_0, \quad e_1 \otimes e_1$$

Very quickly this becomes visually messy.

Dirac notation compresses this into

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

which is **exactly why physicists invented it**.

BASIS STATES AND DIRAC NOTATION

Two-qubit basis

Tensor products become

$$|0\rangle \otimes |0\rangle = |00\rangle$$

$$|0\rangle \otimes |1\rangle = |01\rangle$$

$$|1\rangle \otimes |0\rangle = |10\rangle$$

$$|1\rangle \otimes |1\rangle = |11\rangle$$

These four vectors form a basis of $\mathbb{C}^4$.

A classical computer with two bits can be in **one of four states**.

The state of a quantum computer with two qubits is **any vector in a four dimensional complex space**.

i.e. an infinite state space.



Quantum Gates: Basic Building Blocks

A lot of quantum operations (gates) are analogous to classical gates.

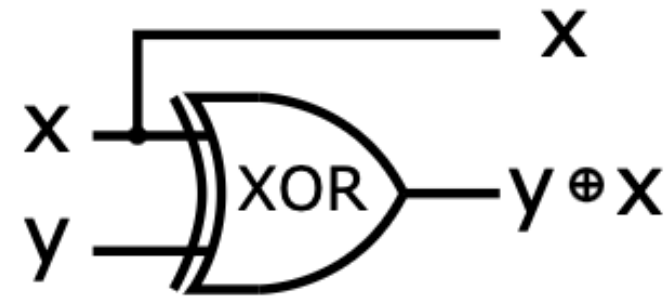
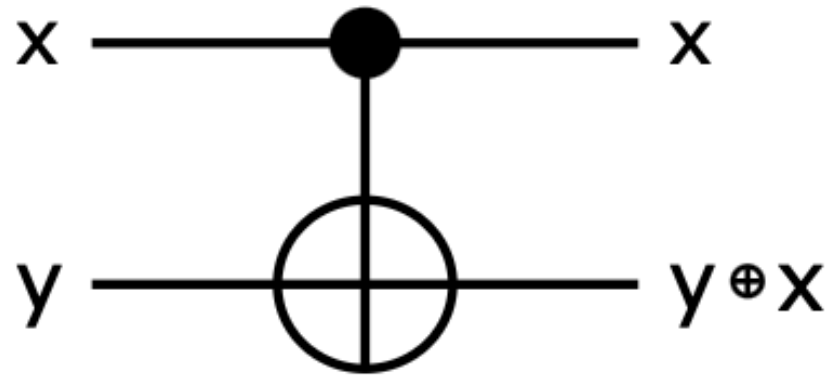
But to take advantage of superposition and coherence we have to make reversible versions.

For example, exclusive or (xor) is not reversible ...

$$\text{xor}(0,0) = 0, \quad \text{xor}(0,1) = 1$$

$$\text{xor}(1,0) = 1, \quad \text{xor}(1,1) = 0$$

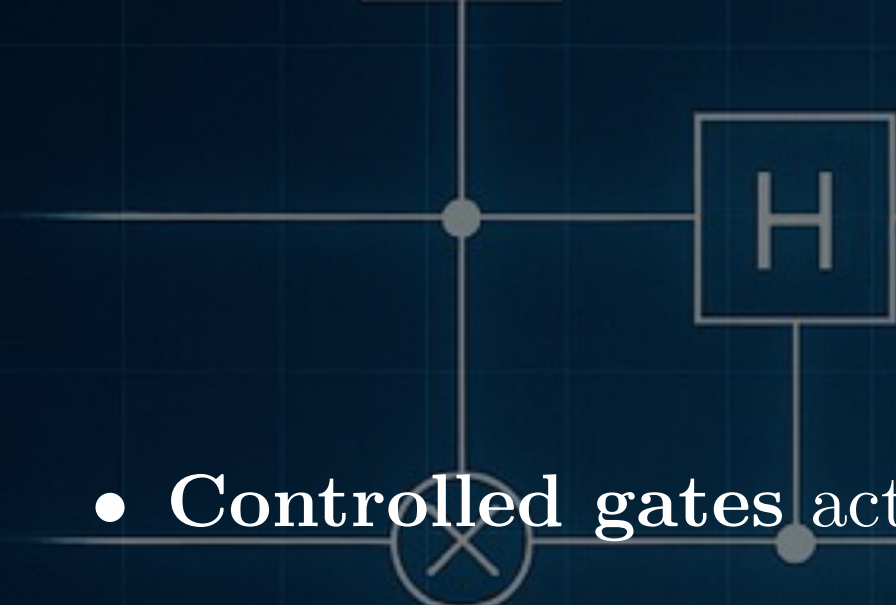
If you get a 1 you don't know which of the inputs was 1



input		output	
x	y	x	y+x
0⟩	0⟩	0⟩	0⟩
0⟩	1⟩	0⟩	1⟩
1⟩	0⟩	1⟩	1⟩
1⟩	1⟩	1⟩	0⟩

input		output	
x	y	x	y+x
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

But control xor is – we output the value of one of the inputs to make it reversible



Controlled Gates and Quantum Flow

- **Controlled gates** act **conditionally** on the control qubit state
- Most common: apply a gate **only if control qubit is in $|1\rangle$**
- Examples:
 - **CNOT**: flips the target qubit if control is $|1\rangle$
 - **CZ**: flips the phase of the target if control is $|1\rangle$
 - **Toffoli (CCNOT)**: flips the target if **two** controls are $|1\rangle$
- Enables entanglement, conditional logic, and quantum interference flow

Building a Superposition Circuit

Let's walk through how we build a quantum state that explores many possibilities in parallel.

We always begin with qubits in the $|0\rangle$ state. These are like classical bits set to 0.

By applying a Hadamard gate to each qubit, we turn each $|0\rangle$ into $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

If we do this to n qubits, we get a superposition over all 2^n possible combinations.

Mathematically, this is written as: $H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$ Each $|x\rangle$ represents a possible bitstring.

Meet the Bell States

Meet the Bell States

The four maximally entangled two-qubit states:

$$\Phi^+ = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$\Phi^- = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)$$

$$\Psi^+ = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle)$$

$$\Psi^- = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

Now we are dealing
with two qubits

$|10\rangle$

Means one qubit is in
state $[0,1]$ and the other
is in state $[1,0]$

- Perfect correlation or anti-correlation when measured
- Basis for quantum communication, teleportation, and error correction

Bell correlations and communication

For the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$, equal-basis outcomes are correlated and each local outcome is random.

Comparing outcomes requires a classical channel. Entanglement alone does not transmit a chosen message.

CNOT and Bell-state preparation

Define the four-by-four CNOT matrix in the declared basis order.

Apply H to one input and then CNOT. Show the joint state after each step and prove that the result does not factor.

Composite-system checkpoints

Compare a product state and an entangled state using amplitudes and joint probabilities.

Add an exercise on operator order and a controlled operation. Reserve software implementation for M_Q.