

M₂: Amplitudes and interference

Real amplitudes, H and Z, and Deutsch's algorithm. No Qiskit in this meeting.

The real-amplitude model

Represent a state by real amplitudes whose squares sum to one. Allowed transformations preserve this norm.

Opposite signs can cancel when alternatives recombine. Measurement probabilities come from squared amplitudes.

FROM WAVES TO QUANTUM CIRCUITS

Everything we saw:

- amplitudes
- phase
- interference

is implemented as: $|\psi\rangle \rightarrow U|\psi\rangle$

THE HADAMARD GATE (H)

We are going to define a fundamental quantum gate. Our goal is for it to:

- spread amplitude evenly across basis states as much as possible
 - make those amplitudes distinguishable through phase
 - use interference between amplitudes to select desired outcomes
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Why?

These properties allow us to exploit quantum parallelism and interference to solve some problems faster than a classical computer can.

THE HADAMARD GATE (H)

Spread amplitude evenly across basis states as much as possible:

Recall from Lecture 6:

The determinant measures how a transformation scales area (generalises to volume in higher dimensions).

Hadamard asked:

What matrix with constrained entries maximises this scaling? (Pure math question, this is long before quantum physics was discovered)

To avoid bias, we choose symmetric constraints:

± 1 (any other constant just rescales the matrix)

Some matrices that maximise area are:

$$\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2, \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$$



Jacques Hadamard
École Normale Supérieure (Paris)
*Résolution d'une question relative aux
determinants* **(Solution to a
problem concerning
determinants), 1893**

THE HADAMARD GATE (H)

We want a gate that:

- spreads amplitude evenly
- introduces relative phase
- preserves length (unitary)

We want a gate that creates an even superposition:

$$G|0\rangle = (|0\rangle + |1\rangle)$$

And a phase difference

$$G|1\rangle = (|0\rangle - |1\rangle)$$



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THE HADAMARD GATE (H)

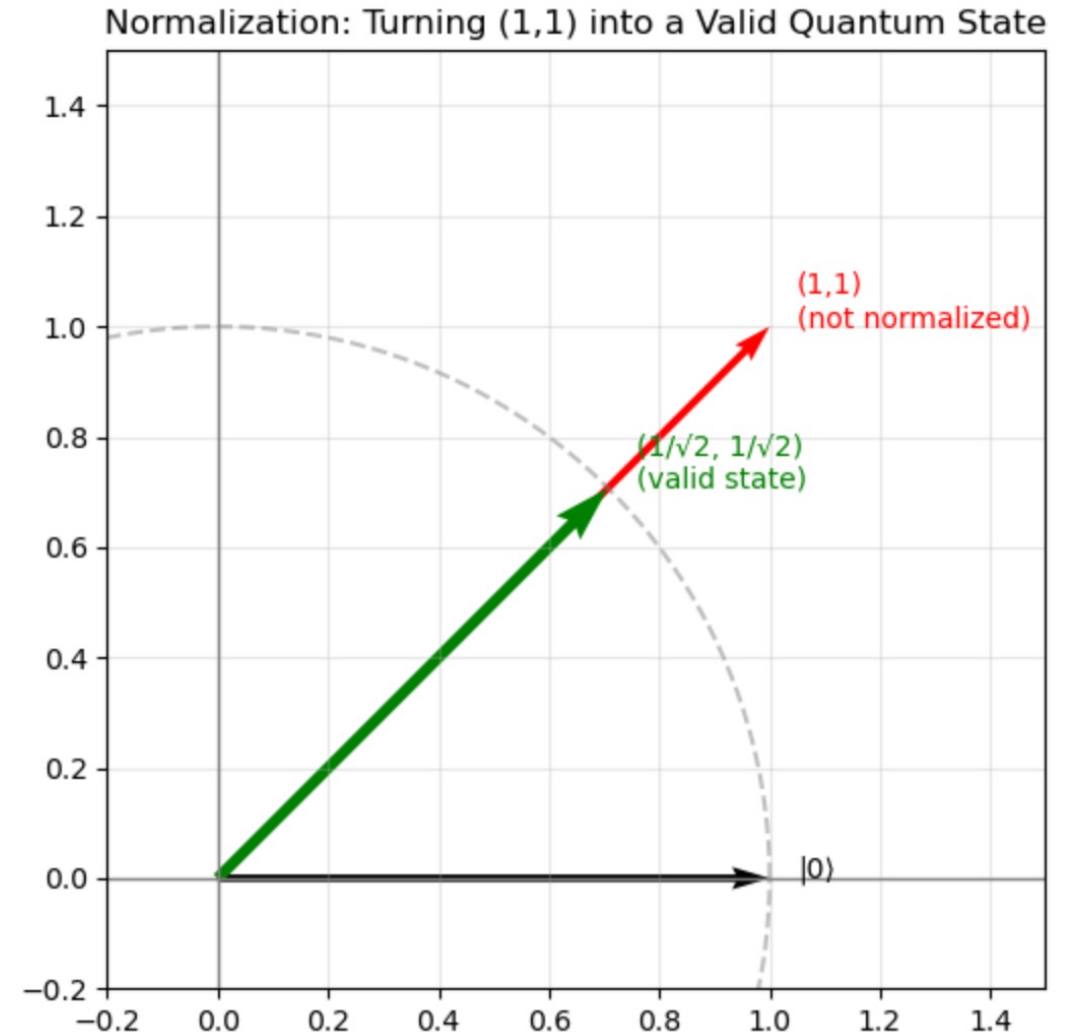
We want a gate that:

- spreads amplitude evenly
- introduces relative phase
- preserves length (unitary)

And to preserve the unit requirement:
we just divide by a length that returns 1:

$$|\psi|^2 = \left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$U|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad U|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$



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David Deutsch

Oxford University

**Quantum theory, the Church–
Turing principle and the universal
quantum computer (1985)**

THE HADAMARD GATE (H)

A simple unitary transformation:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Acts on basis states:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Superposition comes from distributing amplitude across basis states.

Interference arises from the relative phase between those amplitudes.

The Hadamard gate exposes two of the three essential aspects of a quantum computation.

Superposition and Interference. .

QUANTUM NOT/FLIP/PAULI GATE

Example: A Single Qubit Gate

Show a simple matrix acting on

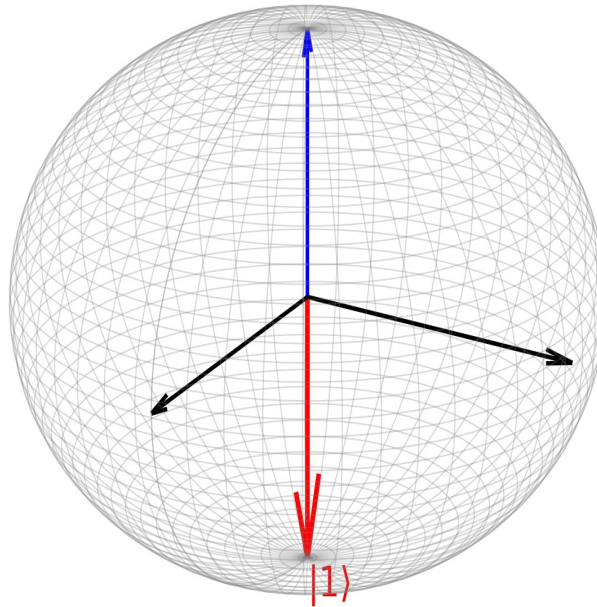
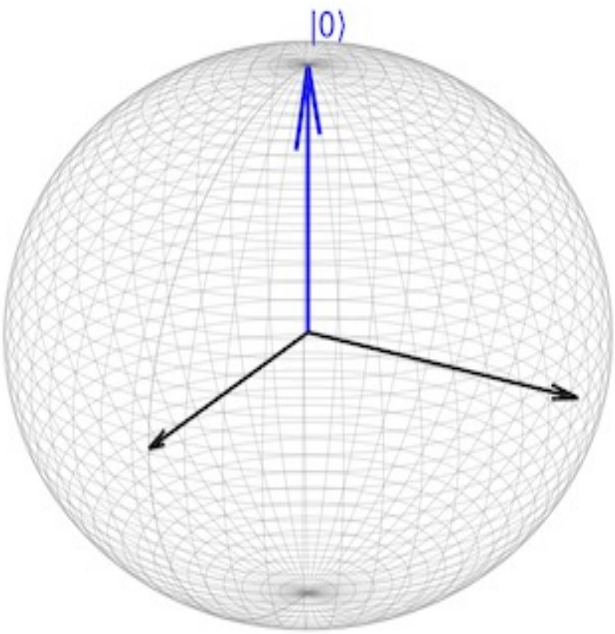
$$|\psi\rangle = (\alpha\beta)$$

Example gate:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Result:

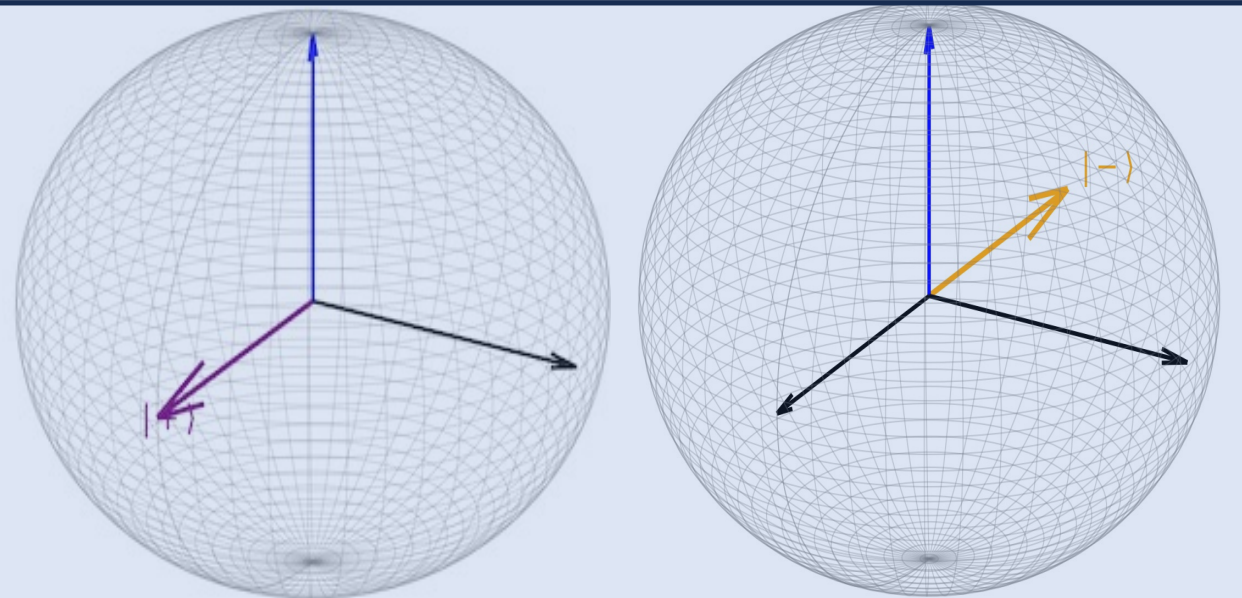
$$X(\alpha\beta) = (\beta\alpha)$$



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Standard Basis



$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hadamard Basis

This is how we put the qubits into superposition

Deutsch's Algorithm

Problem:

Given a function $f : \{0, 1\} \rightarrow \{0, 1\}$, determine whether it is **constant** or **balanced**.

Classical approach: Requires 2 evaluations of f .

Quantum advantage: Requires only 1 quantum evaluation using superposition and interference.

Key Steps:

- Apply Hadamard gates to create superposition.
- Use a quantum oracle to evaluate $f(0)$ and $f(1)$ simultaneously.
- Apply Hadamard to interfere the outputs.
- Measure to determine whether f is constant or balanced.

THE CENTRAL QUESTION

Can one quantum query distinguish a constant function from a balanced one?

Phase kickback places a global property of the oracle into an interference pattern.

Prediction: state what should change in the first experiment and what must remain invariant.

KEY TERMS

oracle

phase kickback

constant/balanced promise

query

MODEL

$U_f|x, y\rangle = |x, y \oplus f(x)\rangle$; one measured input qubit

A sign-interference puzzle

Develop Z as a sign flip and compare H followed by H with H, Z, H .

Trace amplitudes by hand and derive the final measurement probabilities.

Deutsch with an explicit oracle register

Work through all four one-bit Boolean functions using the reversible oracle from M_0 .

Display the four-component state and phase kickback. Introduce the two-register bookkeeping here, then formalize tensor composition in M_3 .