

M₁: Probabilistic computation

Probability vectors, stochastic transformations, and classical sampling baselines.

LIMITATION OF DETERMINISTIC SYSTEMS

Deterministic systems cannot represent:

- randomness
- uncertainty
- incomplete knowledge

But real systems often have:

- noise
 - randomness
 - probabilistic outcomes
-

PROBABILISTIC SYSTEMS

Now the state is:

a probability distribution over
states

For a bit:

$$\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \text{ where } p_0 + p_1 = 1$$

Example probabilistic bit:

$$\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

- 70% chance of 0
- 30% chance of 1

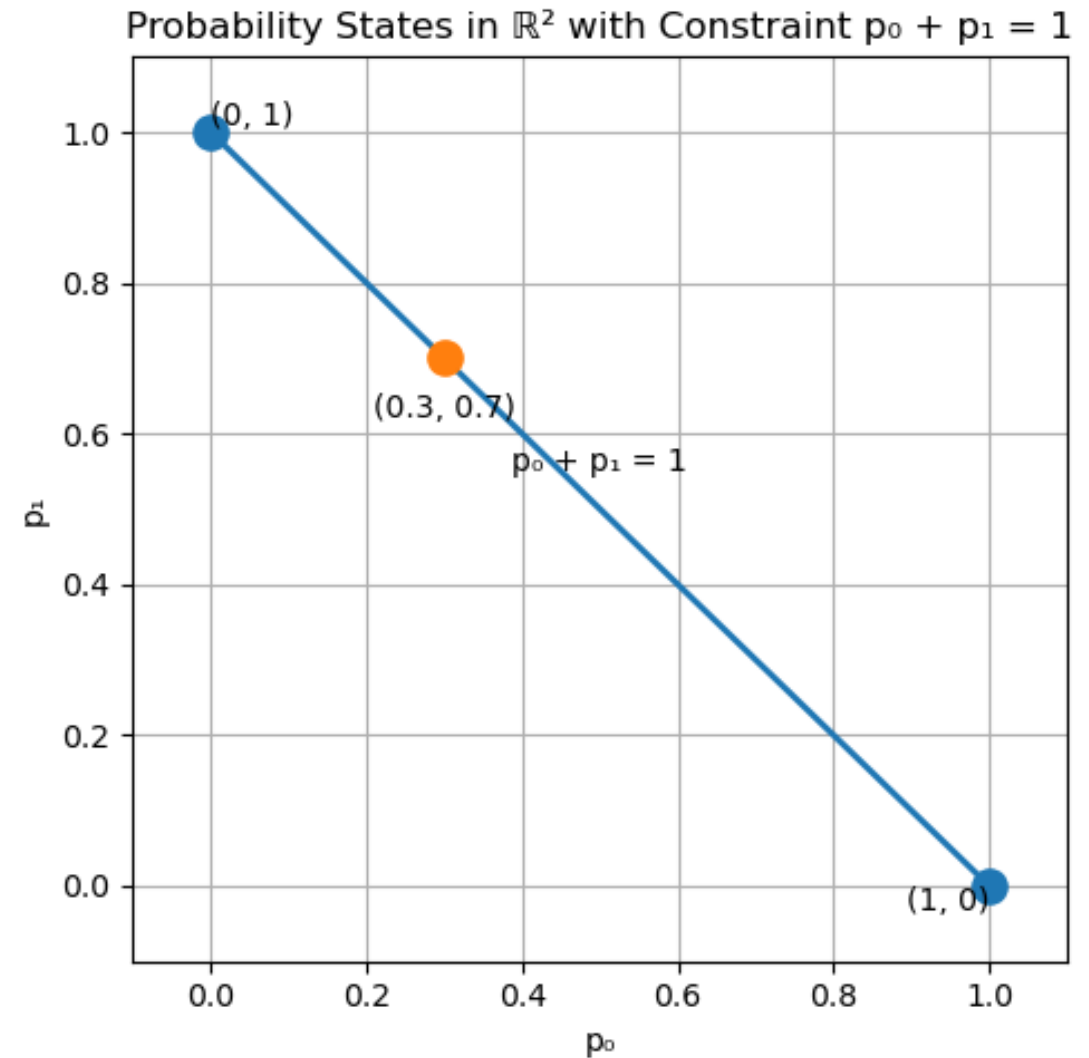
GEOMETRY OF PROBABILITY

- Deterministic states = **basis vectors**

$$0 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 1 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Probabilistic bits are vectors in the space defined by the basis.

They must sum to 1.



FROM DETERMINISTIC TO PROBABILISTIC

- Deterministic systems:
one state goes to one state
- Probabilistic systems:
one state goes to a **distribution over states**

$$\begin{pmatrix} p'_0 \\ p'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

Again the operator is a matrix.

CONSTRAINTS ON PROBABILISTIC TRANSFORMATIONS

We want: $p' = Mp$ to always produce a **valid probability distribution**

Requirements

For all states p : **Non-negativity** $M_{ij} \geq 0$

Probability is conserved

$$\sum_i p'_i = 1$$

A matrix with non-negative entries and columns summing to 1 is called a **stochastic matrix**

Implies:

$$\sum_i M_{ij} = 1 \quad \text{for each column } j$$

Each column is a **probability distribution**

EXAMPLE

Let

$$M = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

and suppose the current state is:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Apply the transformation

$$p' = Mp$$

- Start in state 0 with certainty
- After one step:
 - 80% chance of state 0
 - 20% chance of state 1

PROBABILISTIC STATE TRANSITIONS

Two steps

$$p^{(2)} = Mp' = M(Mp) = M^2p =$$

(in our example) $\begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$

After n steps

$$p^{(n)} = M^n p$$

Definition:

A **Markov chain** is a system where:

- the state is a probability distribution
- evolution is given by a stochastic matrix

$$p^{(n)} = M^n p^{(0)}$$

PROBABILISTIC STATE TRANSITIONS

After n steps

$$p^{(n)} = M^n p$$

In the limit of ∞ steps

$$\text{where: } \lim_{n \rightarrow \infty} M^n p = p^*$$

The pagerank algorithm that made Google the lead search engine was a Markov chain.

$$Mp^* = p^*$$

STATE DISTRIBUTION IN THE LIMIT

Proof

We're looking for the **steady state**:

$$p^{(n)} \rightarrow p^* \quad \text{as } n \rightarrow \infty$$

which satisfies:

$$Mp^* = p^*$$

STATE DISTRIBUTION IN THE LIMIT

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which satisfies:

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Step 1: Solve the fixed-point equation

Let

$$p^* = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x + y = 1$$

and

$$M = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

STATE DISTRIBUTION IN THE LIMIT

We're looking for the **steady state**:

$$p^{(n)} \rightarrow p^* \quad \text{as } n \rightarrow \infty$$

which satisfies:

$$Mp^* = p^*$$

Step 2: Apply $Mp^* = p^*$

$$\begin{pmatrix} 0.8x + 0.3y \\ 0.2x + 0.7y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Step 1: Solve the fixed-point equation

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Step 3: Solve the top equation

$$0.8x + 0.3y = x$$

$$0.8x + 0.3y - x = 0$$

$$(0.8 - 1)x + 0.3y = 0$$

$$-0.2x + 0.3y = 0$$

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Step 3: Solve the top equation

$$\begin{aligned} 0.8x + 0.3y &= x \\ 0.8x + 0.3y - x &= 0 \\ (0.8 - 1)x + 0.3y &= 0 \\ -0.2x + 0.3y &= 0 \end{aligned}$$

Step 4: Solve the bottom equation

$$\begin{aligned} 0.2x + 0.7y &= y \\ 0.2x + 0.7y - y &= 0 \\ 0.2x + (0.7 - 1)y &= 0 \\ 0.2x - 0.3y &= 0 \end{aligned}$$

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Notice that the one equation is the negative of the other.

STATE DISTRIBUTION IN THE LIMIT

Step 5:

$$-0.2x + 0.3y = 0$$

Add $0.2x$ to both sides:

$$0.3y = 0.2x$$

Divide both sides by 0.3 :

$$y = \frac{0.2}{0.3}x = \frac{2}{3}x$$

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Let

$$p^* = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x + y = 1$$

and

$$M = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$$

Solve the top equation

$$0.3y = x$$

$$0.3y - x = 0$$

$$-x + 0.3y = 0$$

$$0.3y = x$$

Step 4: Solve the bottom equation

$$0.2x + 0.7y = y$$

$$0.2x + 0.7y - y = 0$$

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Step 6: Use the fact that probabilities sum to 1

$$x + y = 1$$

Substitute $y = \frac{2}{3}x$:

$$x + \frac{2}{3}x = 1$$

Write x as $\frac{3}{3}x$:

$$\frac{3}{3}x + \frac{2}{3}x = \frac{5}{3}x = 1 \Rightarrow x = \frac{3}{5}$$

Notice that the one equation is the negative of the other.

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Step 6: Use the fact that probabilities sum to 1

Step 7: Plug in and solve

$$x = 0.6$$

Then:

$$y = \frac{2}{3}(0.6) = 0.4$$

Finally,

$$p^* = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$x + y = 1$$

$$x + \frac{2}{3}x = 1 \Rightarrow x = \frac{3}{5}$$

Notice that the one equation is the negative of the other.

WHY?

- **Long-term behaviour**
- Now we know that for this system after many steps, the system spends
- **60% of the time in state 0 and 40% in state 1**

Google PageRank

The [PageRank](#) algorithm uses this same idea:

Model the web as a **Markov process**

Each page is a state

Links define transition probabilities

Then compute the steady state:

Pages with higher long-term probability are ranked higher

Scope of the steady-state example

The preceding calculation concerns the particular transition matrix shown.

A stochastic matrix has a stationary distribution in a finite state space, but repeated evolution need not converge to a unique distribution. Periodic or reducible chains need separate treatment.

Sampling and a classical baseline

Add a sampling puzzle with a declared success criterion and sample budget.

Compare deterministic and randomized query strategies using the same oracle.
Distinguish a distribution from an individual sampled state.