

Unitary dynamics and physical evolution

Global phase, normalization, and unitary evolution. H-gate construction and computational interference move to Q3.

GLOBAL PHASE

A quantum state can be multiplied by a complex phase:

- $|\psi\rangle \rightarrow e^{i\phi} |\psi\rangle$
 - In position form:
 - $\psi(x) \rightarrow e^{i\phi} \psi(x)$
 - every amplitude is multiplied by the same phase
 - relative relationships between amplitudes do not change
 - Two vectors that differ only by global phase are physically identical.
 - Only relative phase can affect measurement outcomes.
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GLOBAL PHASE

Multiplying by a global phase is the simplest length preserving operation:

$$|\psi\rangle \rightarrow e^{i\phi} |\psi\rangle$$

This is a rotation in the complex plane applied to every component.

Length is preserved:

$$|e^{i\phi} \psi|^2 = e^{i\phi} e^{-i\phi} |\psi|^2 = |\psi|^2$$

Global phase defines an equivalence class: which vectors represent the same physical state. This equivalence is what allowed us to reduce the dimensionality of the qubit from a 3-sphere to a 2-sphere (Bloch Sphere).

WHAT MAKES AN EVOLUTION “QUANTUM”?

The state ψ of the computation at $t+1$ is the state at time t after applying a unitary operation U .

$$|\psi_{t+1}\rangle = U|\psi_t\rangle$$

Under the constraint that:

$$|U|\psi\rangle| = |\psi\rangle|$$

I. e. the length* of the resulting vector state is unchanged.

*Remember the length of a complex vector is $\sqrt{\langle\psi, \psi\rangle} = \sqrt{\sum_i \psi_i \bar{\psi}_i}$

WHAT DOES PRESERVING LENGTH IMPLY?

- We have: $|U|\psi\rangle| = |\psi\rangle|$
- Using the inner product: $\langle U\psi, U\psi \rangle = \langle \psi, \psi \rangle$
- Implies: $U^\dagger U = I$

Step 1. Use the definition of the adjoint

- By definition of the adjoint,
$$\langle U\psi, U\psi \rangle = \langle \psi, U^\dagger U\psi \rangle$$
- So the condition becomes
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Step 2. Move everything to one side

Subtract the right hand side:

- $\langle \psi, U^\dagger U\psi \rangle - \langle \psi, \psi \rangle = 0$

Using linearity,

- $\langle \psi, (U^\dagger U - I)\psi \rangle = 0$
- for every $|\psi\rangle$.

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Step 3. Ask what operator could make this true for every vector

For this to hold for all $|\psi\rangle$, the operator inside must be zero: $U^\dagger U - I = 0$ so $U^\dagger U = I$

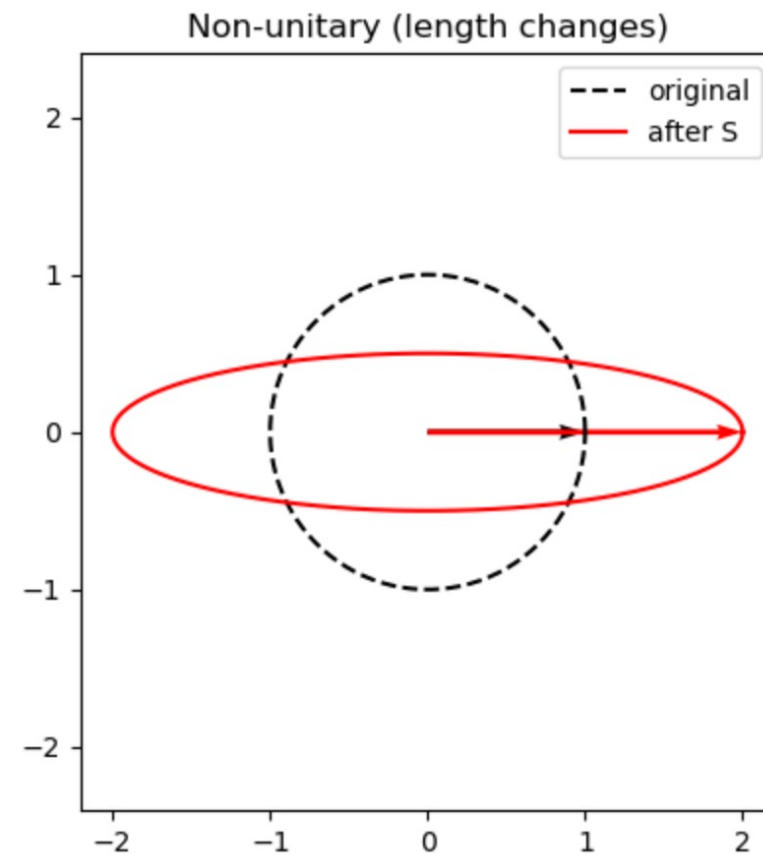
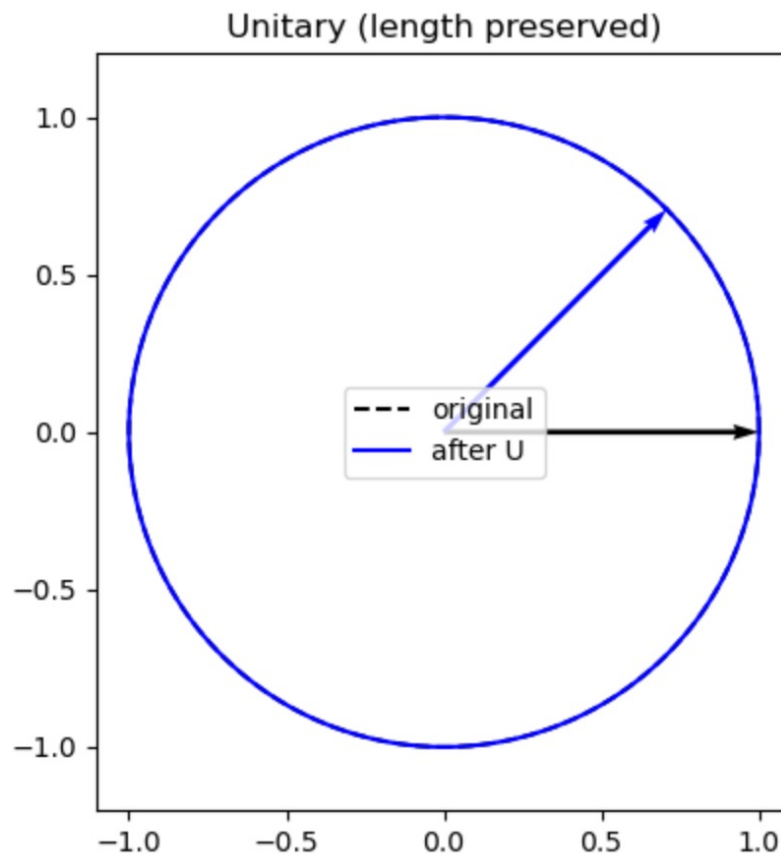
WHAT DOES PRESERVING LENGTH IMPLY?

Applying U does not change inner products with itself so U preserves lengths the only way that can be true for every state is if $U^\dagger$ exactly undoes U

Hence

$$U^\dagger U = I$$

In quantum computing, all allowed operations look like the left panel.



Hamiltonian and time evolution

Derive the connection between the Schrödinger equation and $U(t)$ for a time-independent Hermitian Hamiltonian.

State the assumptions and units. Replace the unfinished derivation in original Lecture 12, slides 41–42.

Energy eigenstates and relative phase

Use a two-level energy spectrum to calculate phase accumulation.

Distinguish a shared global phase from a relative phase observable in a suitable basis.

Physics-to-computation boundary

Finish with a worked physical evolution and its unitary matrix.

Q3 will treat selected transformations as allowed computational operations. Leave circuit software for M_Q.