

Quantum states and measurement probabilities

Amplitudes, Dirac notation, and measurement bases. Computational examples move to Q3.

DEFINING QUANTUM SYSTEMS

- A system has a set of possible states
 - A quantum state is a complex vector
 - States live in a complex vector space
 - Valid states have unit length
 - Global phase does not affect measurement (need to revisit this)
 - Measurements produce probabilistic outcomes
 - Outcomes correspond to basis states
 - Probabilities come from squared amplitudes
 - Evolution is linear and reversible
 - Systems follow strict mathematical rules
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Amplitudes and probabilities

A normalized pure state has complex amplitudes. The probability of a basis outcome is the squared magnitude of its amplitude.

A superposition is a linear combination of basis states. Measurement samples an outcome, rather than revealing every coefficient.

MEASUREMENT AND THE BORN RULE

Measurement produces classical outcomes

Outcomes correspond to basis states

Probabilities come from amplitudes

Square magnitude of each coefficient: $P(i) = |a_i|^2$

Interpretation

- Larger magnitude means higher probability
- Phase does not affect individual probabilities

Key consequence

- Measurement changes the state
- System collapses to observed outcome

Recall that *phase* is the angle of a complex number vector. $z = re^{i\theta}$, r = magnitude (size), θ = **phase (angle)**

DIRAC BRA–KET NOTATION: KET

A state vector is written as a **ket**: $|\psi\rangle$

Dirac notation lets us treat a quantum state as an object in its own right, rather than as a list of coordinates in a particular basis.

Basis states for one qubit are

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(Einstein tensor notation has the same function)

A general qubit state when we care about the coordinate system is: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ *where* α and β are the coordinates in some basis, e. g. $|0\rangle$ and $|1\rangle$.

This is the same state we wrote earlier as a column vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

DIRAC BRA–KET NOTATION: BRA

For every ket: $|\psi\rangle$

there is a corresponding **bra**: $\langle\psi|$

defined as the conjugate transpose

If

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

then

$$\langle\psi| = (\bar{\alpha}, \bar{\beta})$$

Dirac notation lets us treat a quantum state as an object in its own right, rather than as a list of coordinates in a particular basis.

(Einstein tensor notation has the same function)

This is useful because then $\langle\phi|\psi\rangle$ is the dot product of ϕ and ψ .

This doesn't look more useful than writing $\bar{\phi} \cdot \psi$, but when things start to get more complex we will lean on the bra-ket notation.

The name is literally Braket because we will bookend complex expressions with $\langle\psi|$ and $|\psi\rangle$

MEASUREMENT IN BRA-KET NOTATION

AS PROJECTION

- Start from a quantum state $|\psi\rangle$
- Let's give it coordinates with basis vectors $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- We can extract the state amplitudes with:

- Apply the Born rule

$$P(0) = |\langle 0 | \psi \rangle|^2$$

$$P(1) = |\langle 1 | \psi \rangle|^2$$

Measurement picks out the component of the state along a basis direction, because basis directions are the only thing we can observe.

What can we observe about quantum systems?

A quantum state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

contains:

amplitudes (complex numbers)
phases
superposition structure

But *you cannot directly observe* α or β

Only measurement outcomes: $|0\rangle$ and $|1\rangle$
with probabilities: $|\alpha|^2, |\beta|^2$

Measurement makes a particular basis “real” by projecting the state onto it, but the quantum state itself is richer than any single basis description.

THE GEOMETRY OF A QUBIT: BLOCH SPHERE

Start from the general state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

with

$$|\alpha|^2 + |\beta|^2 = 1$$

Re-parameterise the state

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

θ controls **the relative contribution of each basis state to $|\psi\rangle$** (i.e. how much of 0 and 1 by setting the magnitudes of the amplitudes)

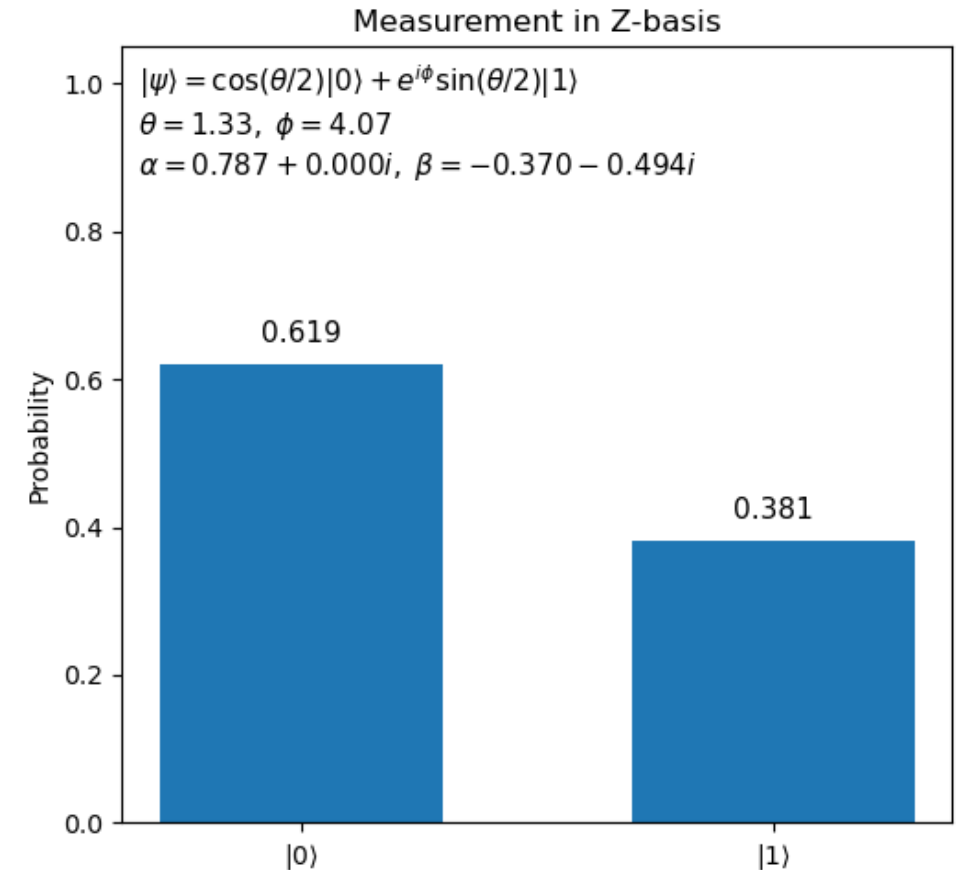
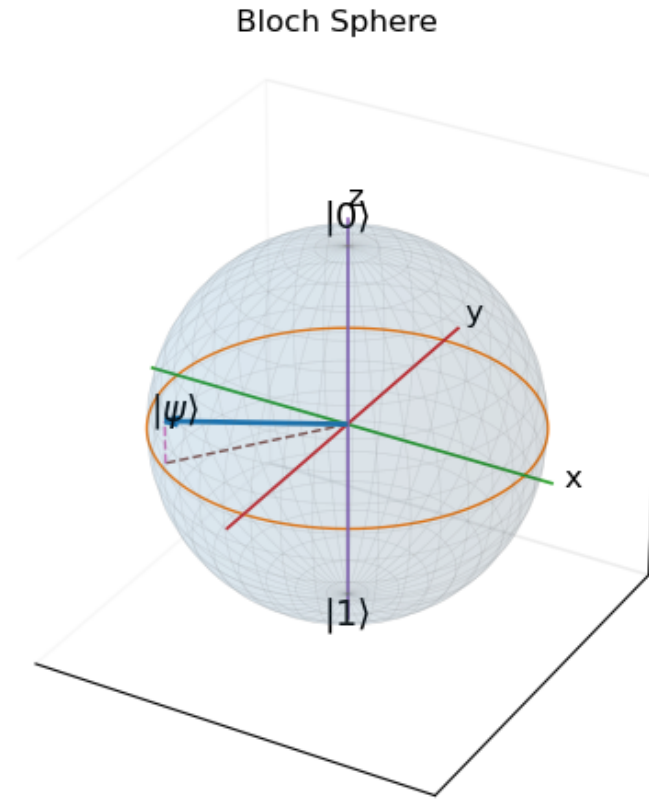
ϕ controls the **relative phase**

Together they let us set/describe the state of a qubit, i.e. a unique point on the surface of the bloch sphere.

MEASUREMENT ON THE BLOCH SPHERE

Measurement in the computational basis

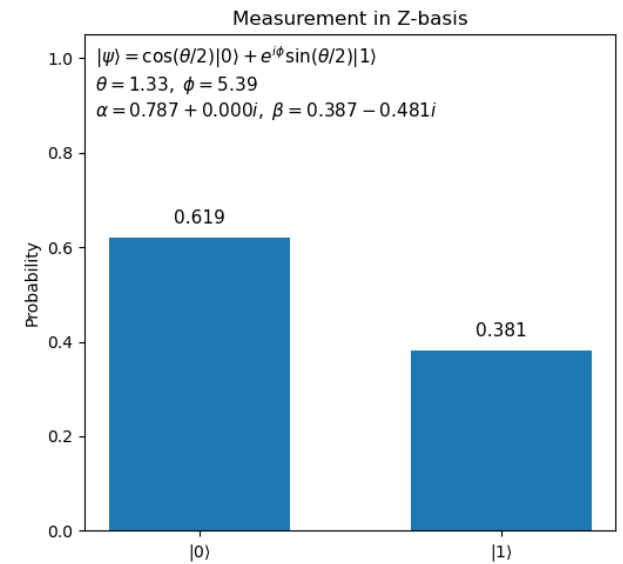
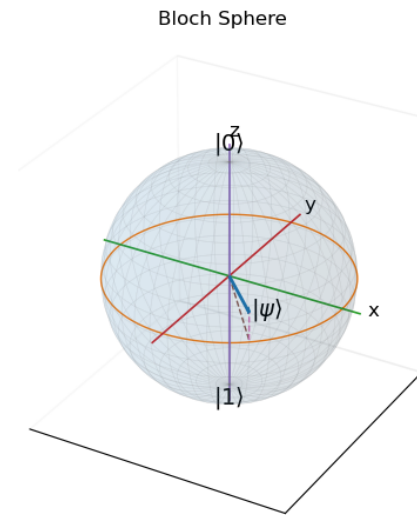
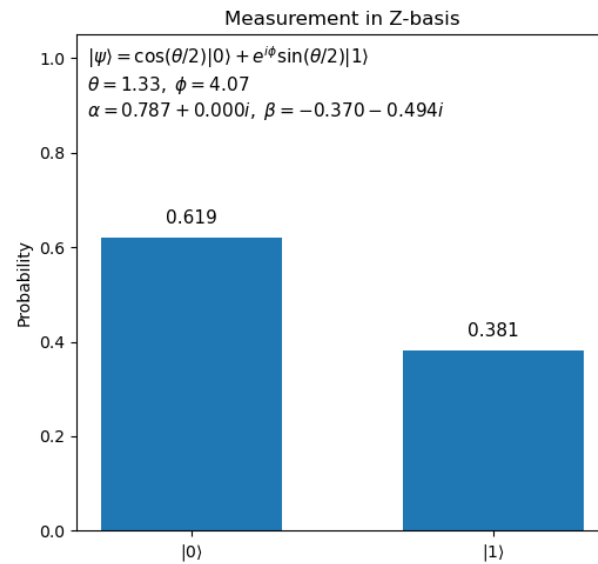
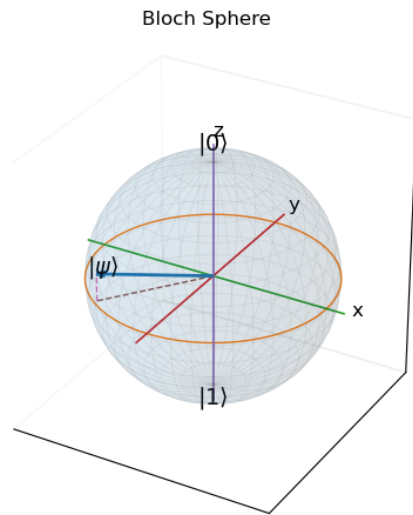
- Measure in the $\{|0\rangle, |1\rangle\}$ basis
- This corresponds to the **Z-axis** on the Bloch sphere



CHANGING THE MEASUREMENT BASIS

- In Z-basis:
- ϕ has **no effect on measurement outcomes**

Z-basis $\rightarrow$ projection onto Z-axis



CHANGING THE MEASUREMENT BASIS

The X-basis states

$$\begin{aligned} |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \end{aligned}$$

Measurement in X-basis

$$P(+)=|\langle +|\psi\rangle|^2$$

$$P(-)=|\langle -|\psi\rangle|^2$$

$|0\rangle \rightarrow$ north pole, pure $|0\rangle$

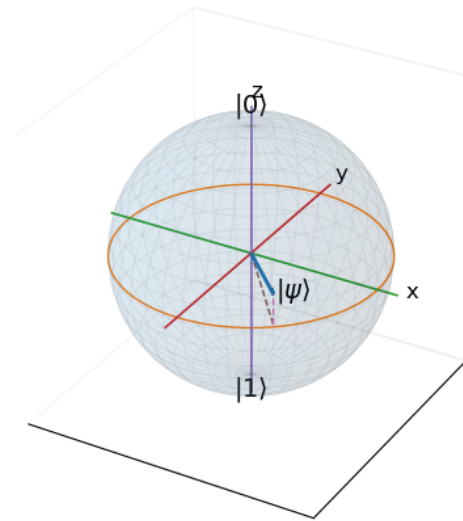
$|1\rangle \rightarrow$ south pole, pure $|1\rangle$

$|+\rangle \rightarrow$ +X direction, $|0\rangle$ and $|1\rangle$ same magnitude and phase

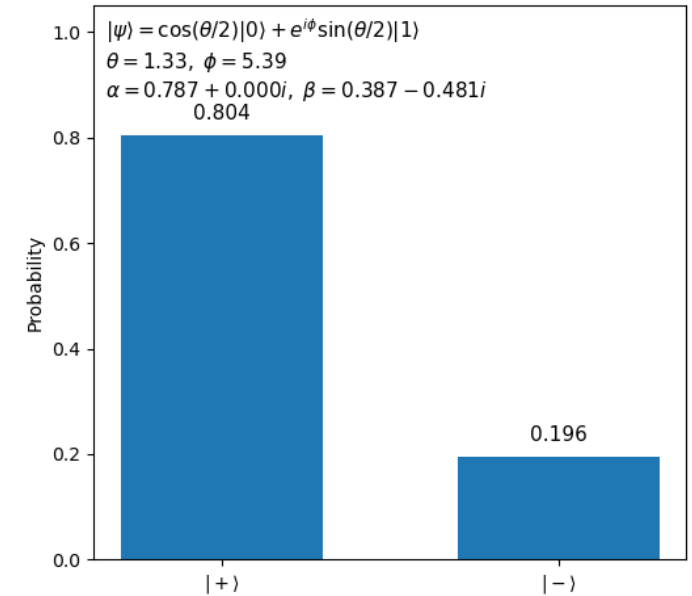
$|-\rangle \rightarrow$ -X direction,

$|-\rangle$ is destructive interference and $|+\rangle$ is constructive

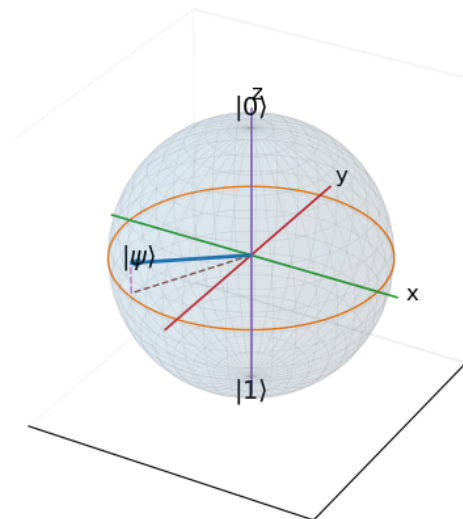
Bloch Sphere



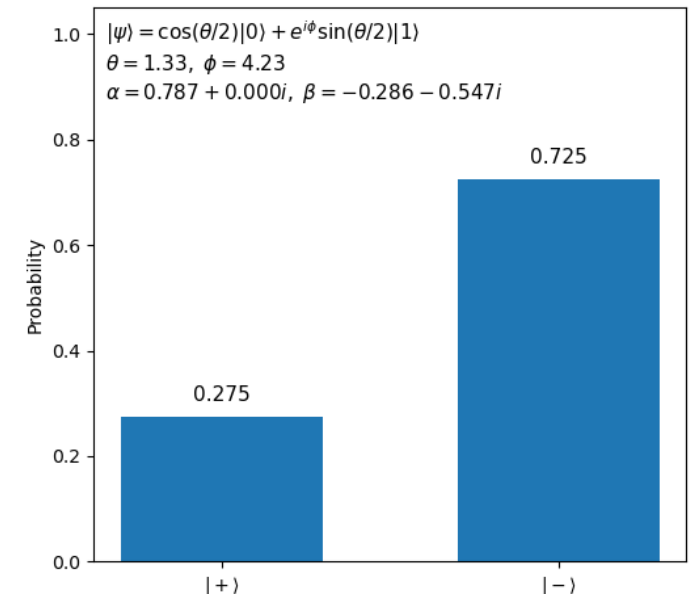
Measurement in X-basis



Bloch Sphere



Measurement in X-basis



WHAT DOES THIS MEAN?

Think of these as just **vectors**:

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now build the two states

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Think of two arrows:

$|+\rangle$: both components point “up” $\rightarrow$ reinforce

$|-\rangle$: one up, one down $\rightarrow$ partially cancel in some directions

FOUR USEFUL QUBIT STATES

In the Z basis:

A: $|0\rangle$: $\theta = 0$

B: $|1\rangle$: $\theta = \pi$

In the X basis:

C: $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$: $\theta = \frac{\pi}{2}$, $\phi = 0$

D: $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$: $\theta = \frac{\pi}{2}$, $\phi = \pi$

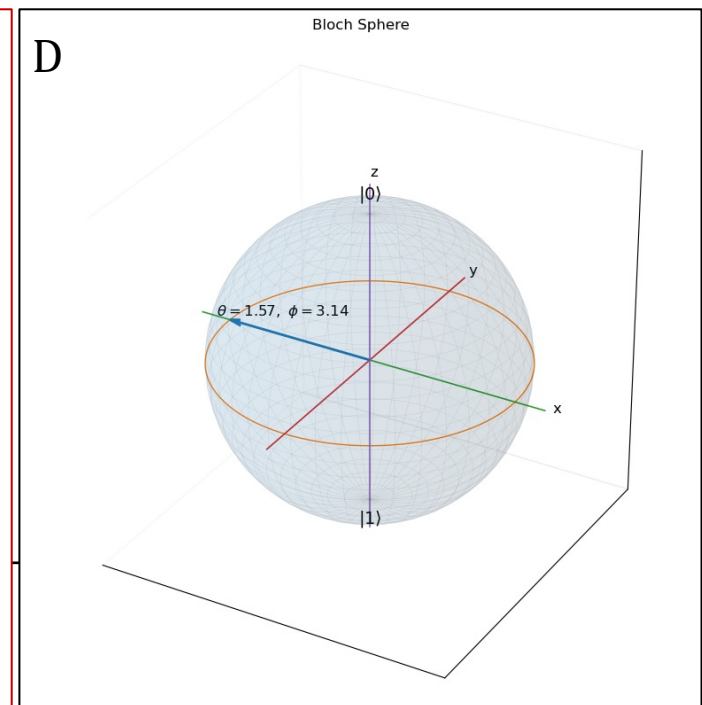
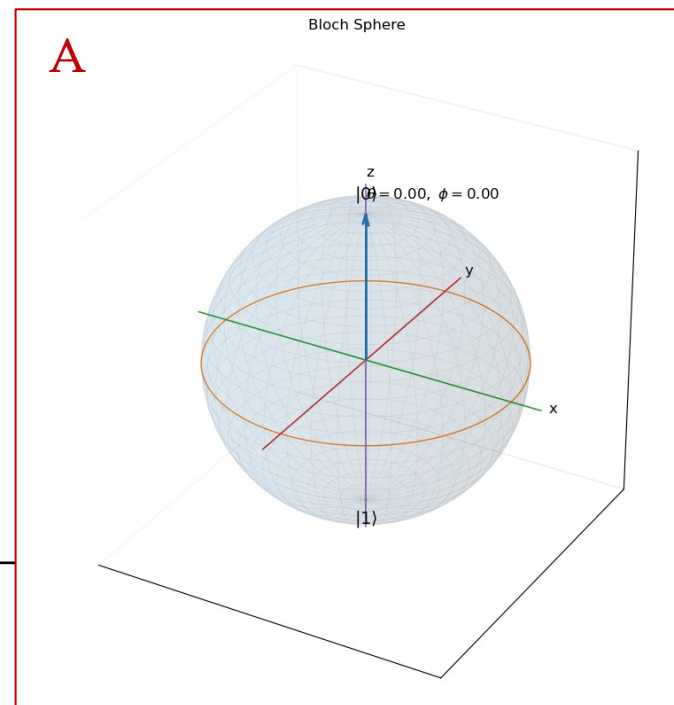
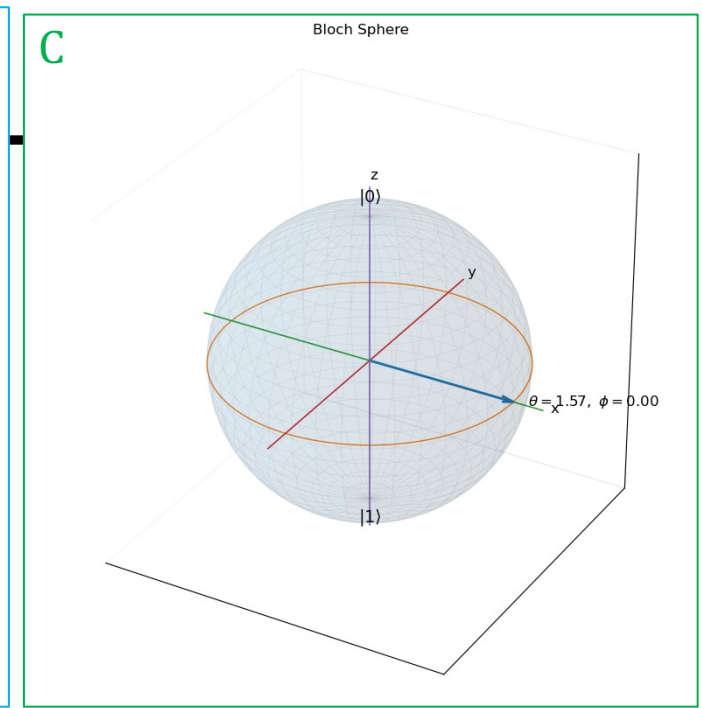
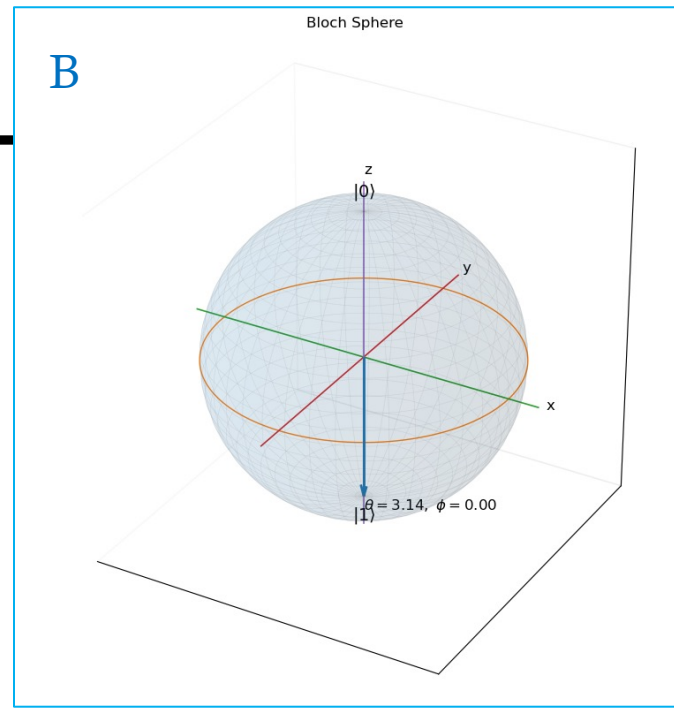
Z basis: “Are you 0 or 1?”

X basis: “Are you + or -?”

They are orthonormal pairs

$|0\rangle \perp |1\rangle$

$|+\rangle \perp |-\rangle$



Preparation versus statistical mixture

Develop one physical preparation of a superposition and one mixture with the same probabilities in a chosen basis.

Use a second measurement basis to distinguish them.

A complete Born-rule example

Specify a normalized state and a measurement basis. Calculate every outcome probability and the post-measurement state.

Include a student checkpoint with a different basis.