



BASIS PRODUCTS AND NORMS

Lecture 5

RECAP

We have built the mathematical setting for quantum computation:

- Complex vector spaces $\rightarrow$ allow superposition
 - Inner products $\rightarrow$ give length, angle, probability
 - Conjugation $\rightarrow$ fixes geometry in complex space
 - Completeness $\rightarrow$ ensures limits stay inside the space
-

RECAP

When you write quantum code, you are:

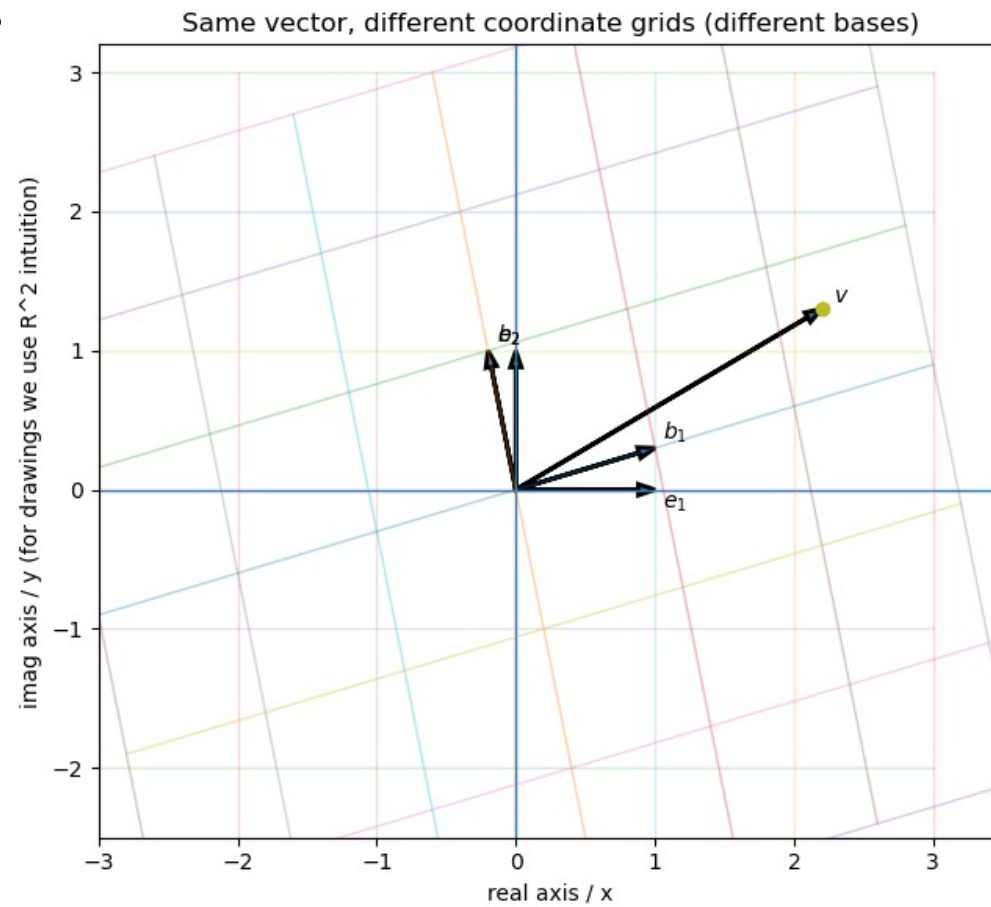
- Preparing vectors
- Combining basis states
- Applying length-preserving transformations
- Computing inner products to obtain probabilities

To understand quantum computation, you must understand the geometry of $\mathbb{C}^n$.

FROM HILBERT SPACES TO COORDINATES

MAKING HILBERT SPACE COMPUTATIONAL

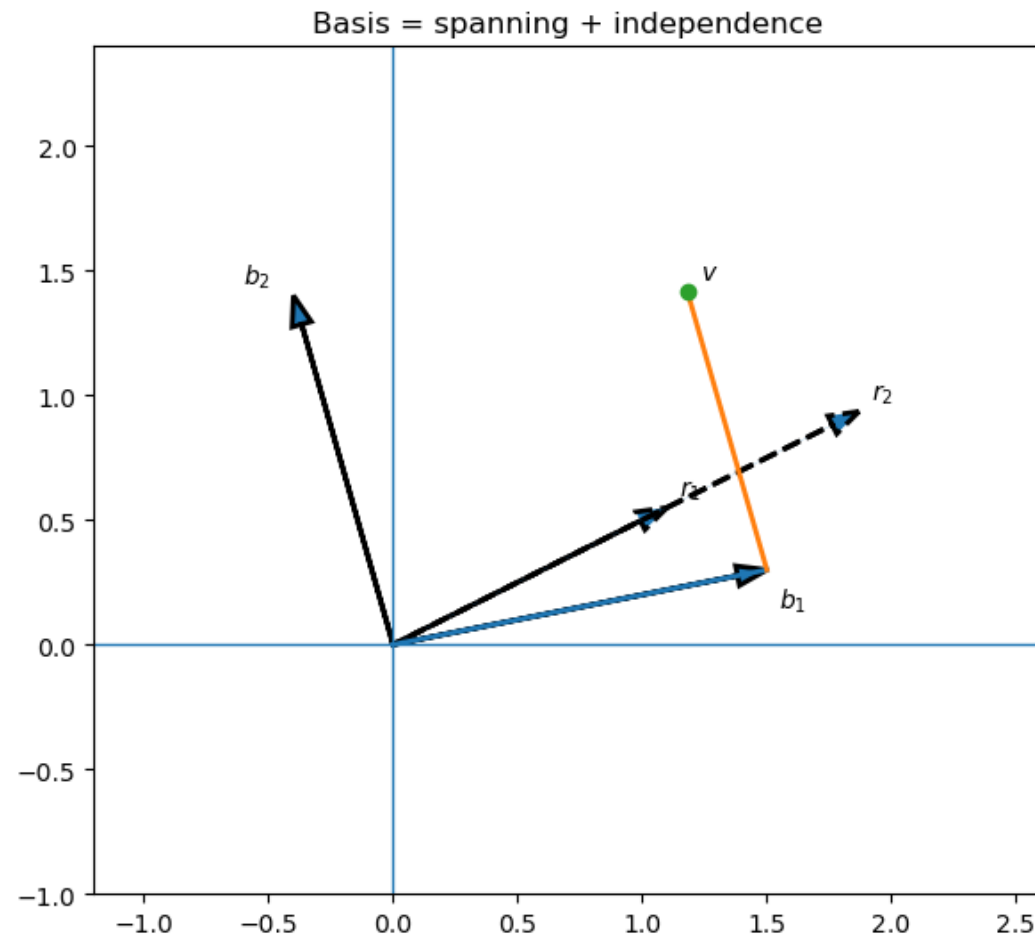
- A vector is a geometric object, independent of any coordinate system.
- Coordinates are the numbers you get after choosing a basis.
- Same vector, different basis, different coordinates.
- Goal today: move between geometry (vectors) and computation (coordinates and matrices).



WHAT IS A BASIS?

DEFINITION OF BASIS IN A COMPLEX VECTOR SPACE.
LINEAR INDEPENDENCE AND SPANNING.

- A **basis** is a set of vectors that lets us describe every vector in the space.
- Two requirements:
- **Spanning:** every vector can be written as a linear combination of the basis vectors
- **Independence:** the basis vectors are not redundant
- In $\mathbb{R}^n$ (and similarly in $\mathbb{C}^n$):
- any two non parallel vectors form a basis
- The dashed line is redundant and does not help form a basis.



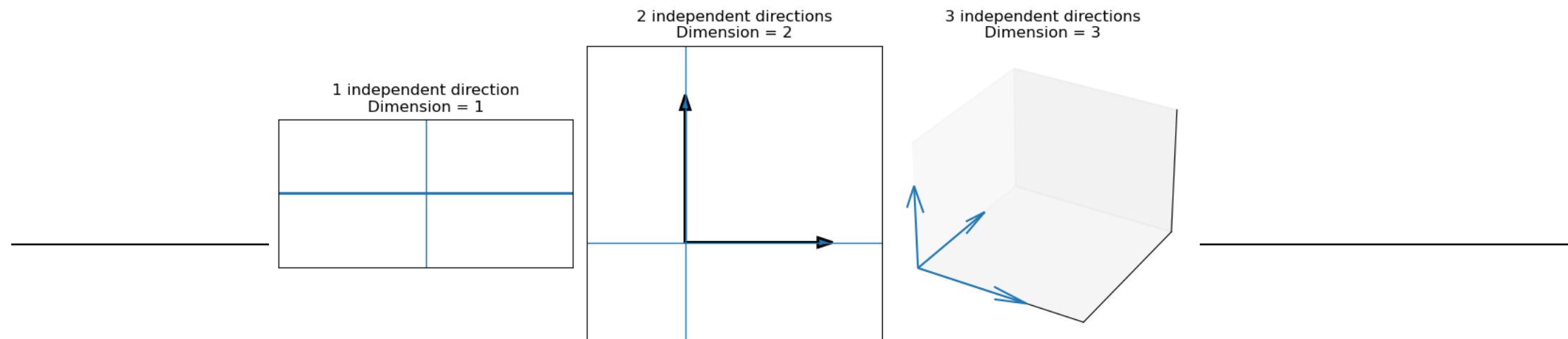
DIMENSION AND DEGREES OF FREEDOM

FINITE-DIMENSIONAL HILBERT SPACES.

CONNECTION TO QUBITS: $\mathbb{C}^{2^N}$.

- The **dimension** of a vector space is the number of vectors in any basis.
- In $\mathbb{R}^2$: dimension = 2
- In $\mathbb{C}^2$: dimension = 2
- In $\mathbb{C}^n$: dimension = n

In quantum computing:
A single qubit lives in $\mathbb{C}^2$
Two qubits live in $\mathbb{C}^4$
n qubits live in $\mathbb{C}^{2^n}$

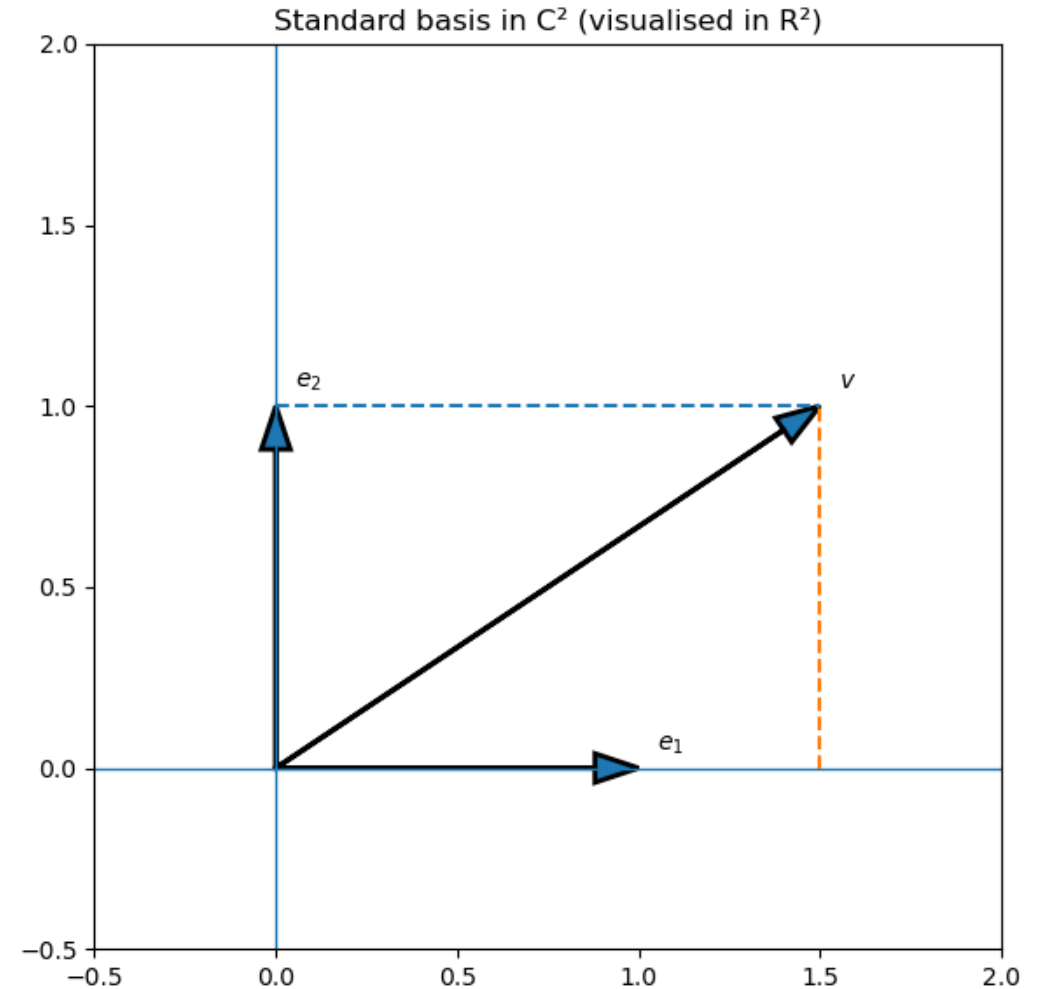


COORDINATES RELATIVE TO A BASIS

EXPRESSING VECTORS AS COORDINATE LISTS. COLUMN VECTOR REPRESENTATION.

The Standard Basis in $\mathbb{C}^n$ In $\mathbb{C}^n$, the standard basis is:

- These basis vectors become computational states.
- For a single qubit:



CHANGE OF BASIS

A VECTOR EXISTS INDEPENDENT OF THE COORDINATE SYSTEM,
THE CHOICE OF COORDINATE SYSTEM DETERMINES THE NUMBERS WE USE TO REPRESENT A VECTOR

If $v = \sum_i v_i e_i$ and $v = \sum_i v'_i b_i$
then the coordinate vectors are related
by a matrix:

$$[v]_b = P^{-1}[v]_e$$

where P is the matrix whose columns
are the new basis vectors expressed in
the old basis.

Changing basis = multiplying by a matrix.

This is why matrices represent
transformations.

In quantum computing:

- Changing measurement basis
- Rotating state representations
- Switching between computational and eigenbases

are all change-of-basis operations.

CHANGE OF BASIS

We change basis to describe vectors in directions that reveal hidden structure.

Let

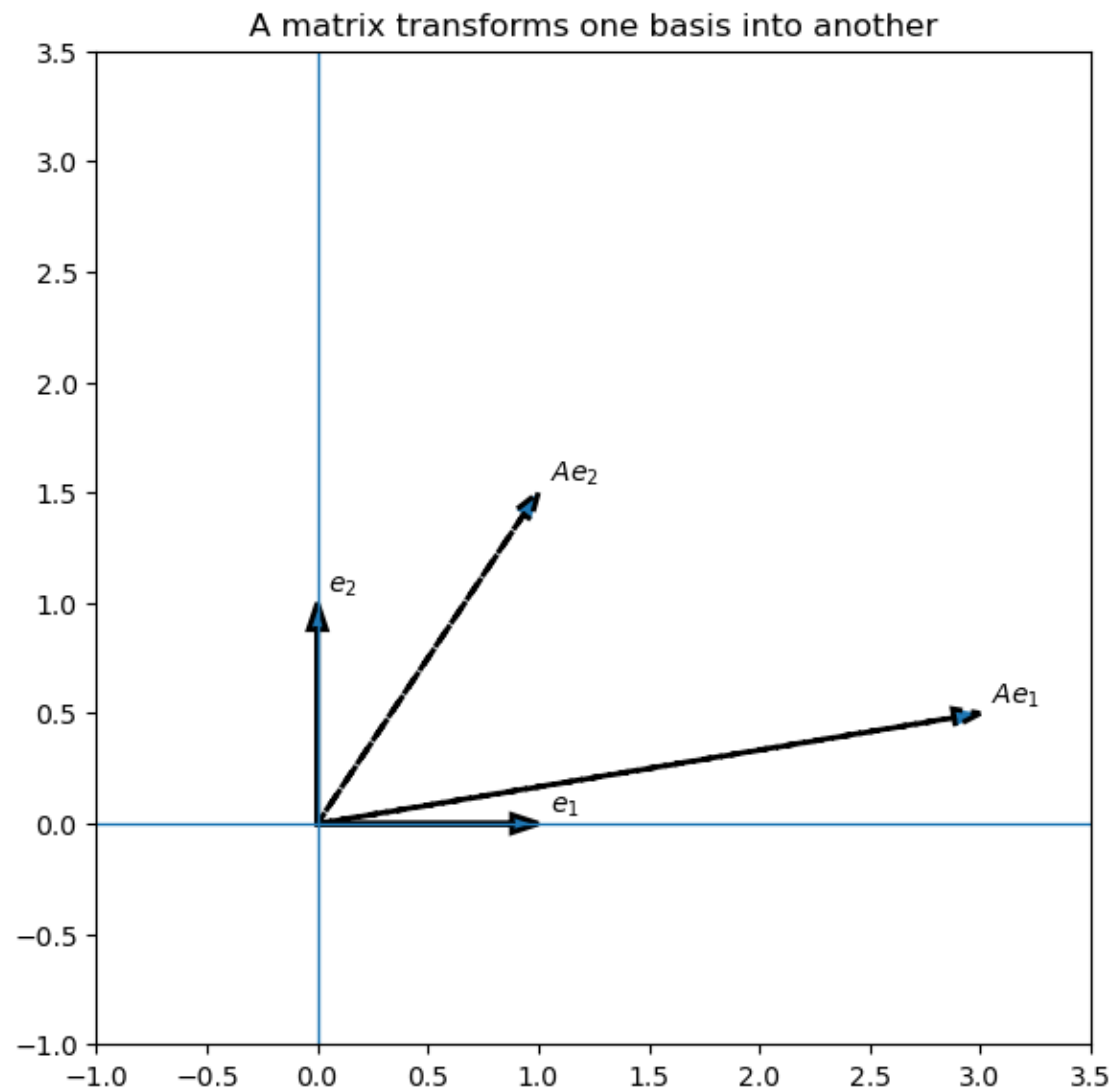
$$A = \begin{pmatrix} 3 & 1 \\ 0.5 & 1.5 \end{pmatrix}, \quad e_1 = (10), \quad e_2 = (01).$$

Then

$$Ae_1 = (30.5), \quad Ae_2 = (11.5).$$

Therefore

$$A = [Ae_1 \ Ae_2]$$



INNER PRODUCTS IN COORDINATES

HOW BASIS CHOICE AFFECTS INNER PRODUCT.

In an orthonormal basis,

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

the inner product is

$$\langle u, v \rangle = \sum_{k=1}^n u_k \overline{v_k}.$$

What This Means

- Multiply corresponding coordinates
- Conjugate the second vector
- Add the results

The geometry is encoded in coordinates.

Why This Matters

This formula lets us compute:

- Length: $|v_k|^2$
- Orthogonality: $\langle u, v \rangle = 0$
- Angles and projections

Everything quantum programs compute relies on this formula.

NORMS AND LENGTH IN COORDINATES

NORMALIZATION OF QUANTUM STATES.

For a vector

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n,$$

the norm is

$$|v| = \sqrt{\langle v, v \rangle} = \sqrt{\sum_{k=1}^n |v_k|^2}.$$

Interpretation

- Square each coordinate's magnitude
- Add them
- Take the square root

This is the Pythagorean theorem in $\mathbb{C}^n$.

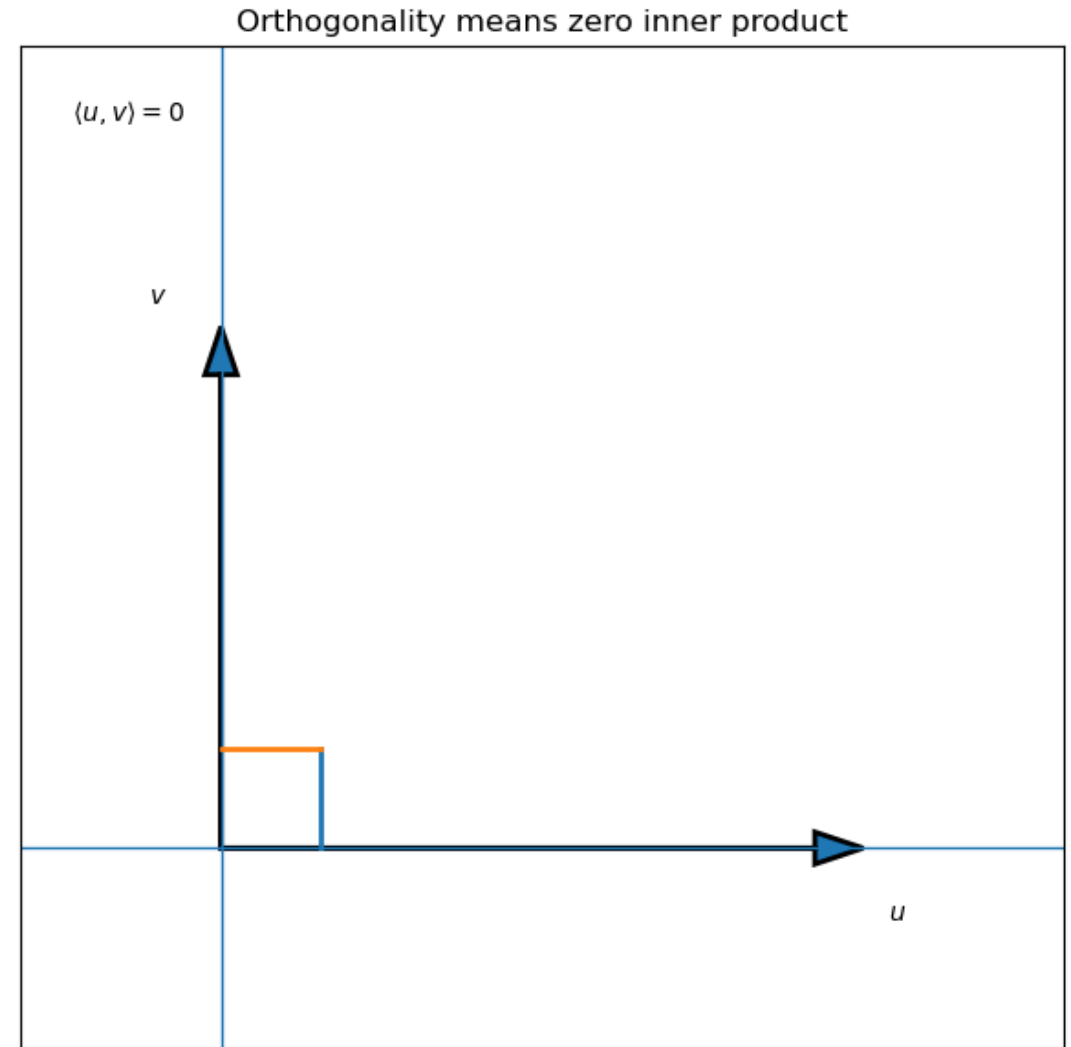
ORTHOGONALITY

QUANTUM MEANING: DISTINGUISHABLE STATES.

- Two vectors are **orthogonal** if their inner product is zero:

$$\langle u, v \rangle = 0$$

- In an orthonormal coordinate system, orthogonality means the vectors share no “overlap”.
- In quantum computing, orthogonal states are perfectly distinguishable by measurement.



ORTHONORMAL BASES

DEFINITION AND PROPERTIES. WHY THEY SIMPLIFY INNER PRODUCTS.

A basis $\{e_1, \dots, e_n\}$ is **orthonormal** if:

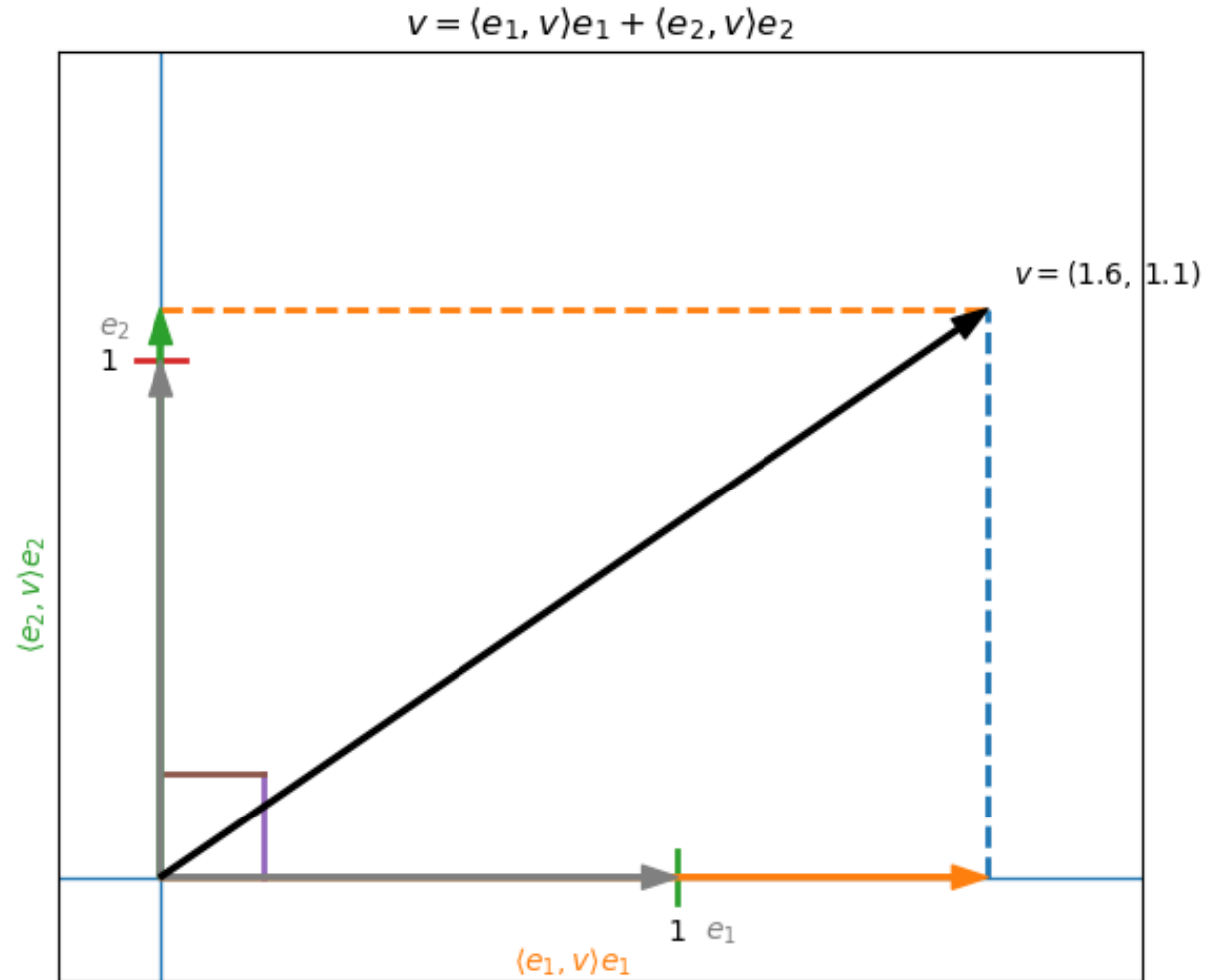
This means:

$$\begin{aligned} |e_i| &= 1 \text{ (unit length)} \\ \langle e_i, e_j \rangle &= 0 \text{ for } i \neq j \text{ (orthogonal)} \end{aligned}$$

Why we like orthonormal bases:

$$v = \sum_{k=1}^n \langle e_k, v \rangle e_k$$

Coordinates become inner products.



ORTHONORMAL BASES

You might think I am just showing you the same old axis you have been using since high-school. I.e. x, y, z

But you learned that coordinates come first. Then vectors live inside coordinates.

What I am showing you is that the vectors define the coordinate system, not the other way around.

The inner product defines geometry.
Coordinates are the result of that geometry.

In quantum computing the coordinates are defined by inner products and the coordinates are amplitudes.

$v = (1.6, 1.1)$

EXPANDING VECTORS IN AN ORTHONORMAL BASIS

THIS IS CRUCIAL FOR MEASUREMENT AND SPECTRAL DECOMPOSITION.

If $\{e_1, \dots, e_n\}$ is orthonormal, then

$$v = \sum_{k=1}^n \langle e_k, v \rangle e_k$$

later, we will reinterpret the same expansion physically

- Inner products extract coordinates.
- Coordinates multiply basis directions.
- The vector is reconstructed from its components.

The inner product extracts the coordinate of v in direction e_k .

LINEAR TRANSFORMATIONS IN A BASIS

INTRODUCE OPERATORS. MATRIX REPRESENTATION RELATIVE TO A BASIS.

A linear transformation $A : V \rightarrow V$ satisfies:

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$$

Algebraic properties

- Linear maps preserve addition.
- Linear maps preserve scalar multiplication.
- They move vectors while preserving structure.

Determined entirely by its action on a basis.

Geometric properties

A linear transformation can:

- Rotate
- Stretch
- Reflect
- Shear

But it always sends lines through the origin to lines through the origin.

MATRICES AS LINEAR MAPS

MULTIPLICATION AS COMPOSITION OF MAPS.

A linear map is a function that takes vectors in and produces new vectors out, in a way that respects addition and scaling.

Once a basis is chosen, every linear transformation
 $A : V \rightarrow V$
is represented by a matrix.

The matrix does not *define* the map.
It represents the map relative to a basis.

Suppose

$$A = \begin{pmatrix} 3 & 1 \\ 0.5 & 1.5 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then

$$Av = \begin{pmatrix} 3 & 1 \\ 0.5 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ 0.5x + 1.5y \end{pmatrix}.$$

If $A: V \rightarrow V$ and $B: V \rightarrow V$,
then AB means:

$$AB(v) = A(B(v)).$$

Matrix multiplication is **composition of linear maps**.

A linear map cannot: Shift the origin, bend space, warp curves. It can only: Rotate, Stretch, Reflect, Shear

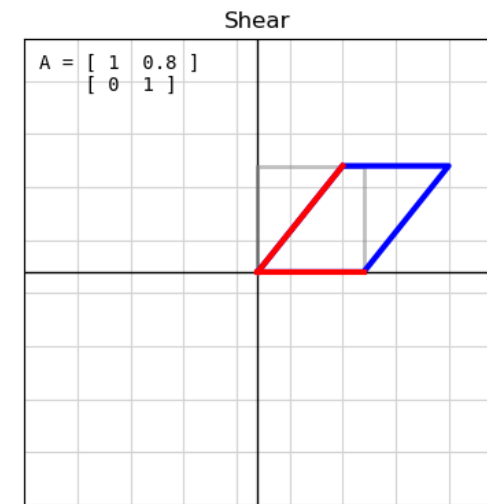
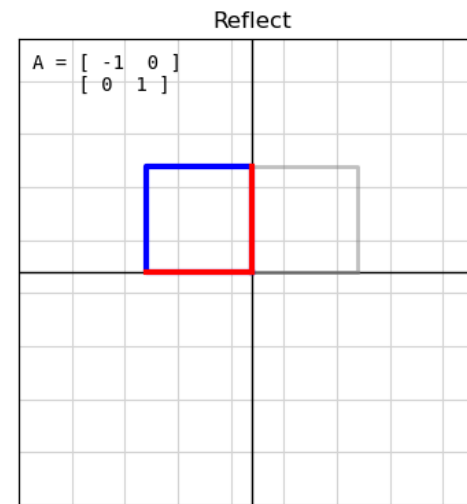
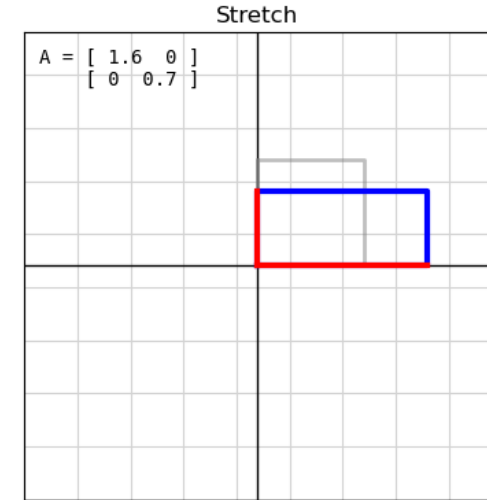
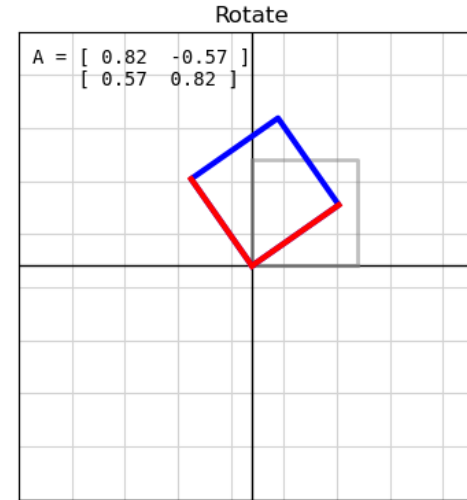
MATRICES AS LINEAR MAPS

MULTIPLICATION AS COMPOSITION OF MAPS

A linear map can Rotate,
Stretch, Reflect, Shear

It will turn out that quantum
computing depends on linear
maps. Non-linear
transformations break quantum
computing.

Linear Maps: Geometry of 2x2 Matrices

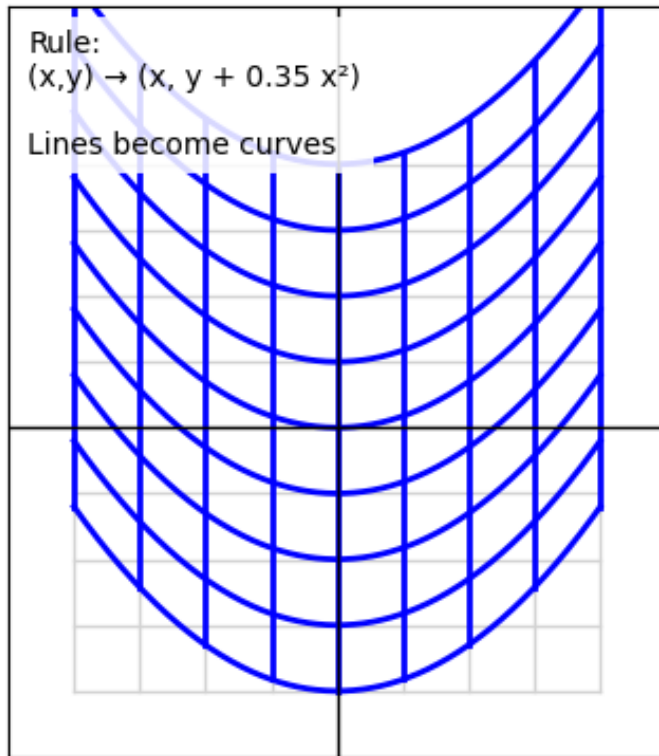


MATRICES AS LINEAR MAPS

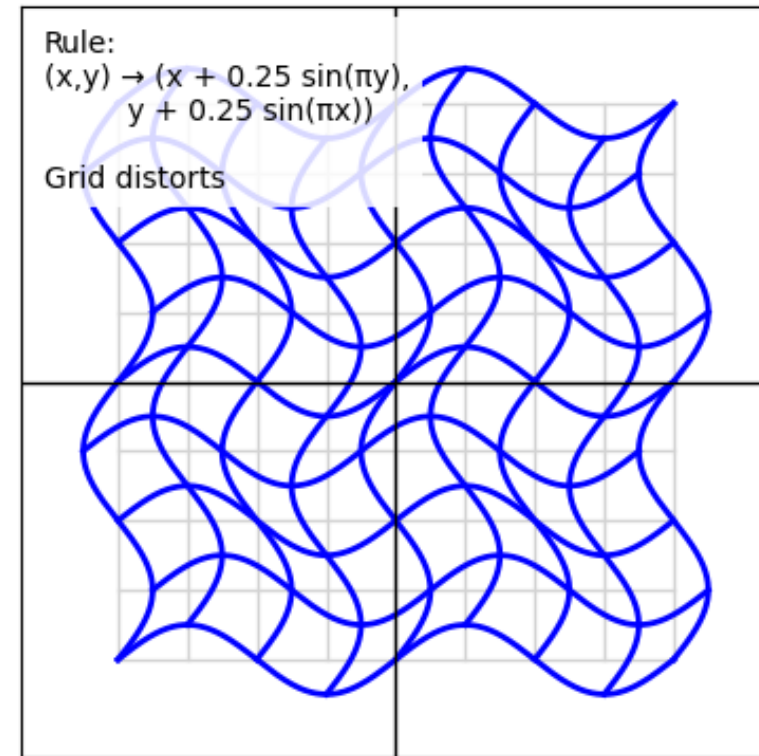
MULTIPLICATION AS COMPOSITION OF MAPS.

What Linear Maps Cannot Do

Not Linear: bending space



Not Linear: warping space



MATRIX TRANSPOSE: A^T

MATH 314 and 321 : Linear Algebra

A matrix is a rectangular array of numbers.

Example: $A = \begin{pmatrix} 1.5 & 0.4 \\ 0.7 & 1 \end{pmatrix}$

The **transpose** of a matrix is formed by flipping rows and columns.

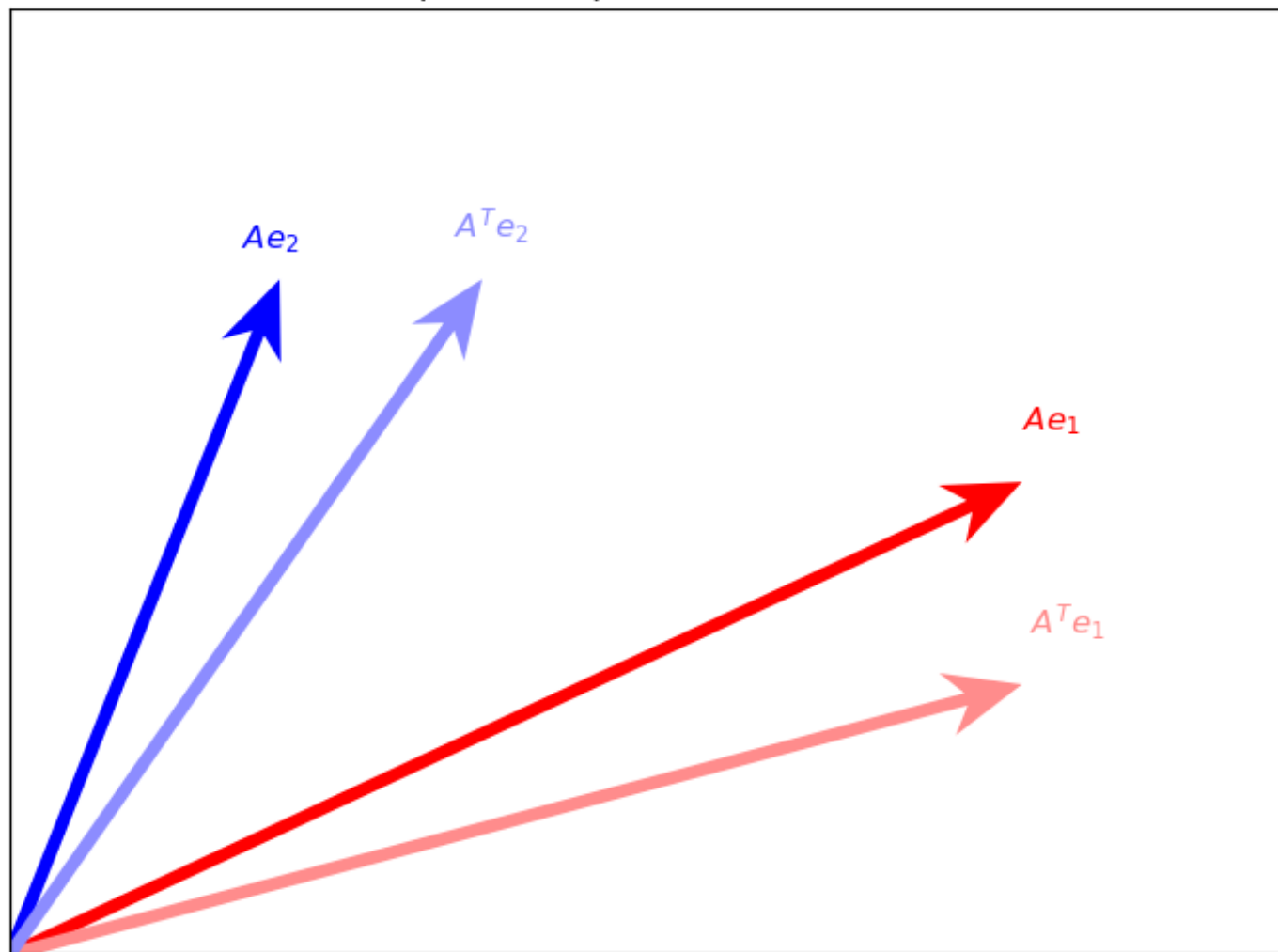
$$A^T = \begin{pmatrix} 1.5 & 0.7 \\ 0.4 & 1 \end{pmatrix}$$

Rows become columns. Columns become rows.

If A_{ij} is the entry in row i , column j ,
Then $(A^T)_{ij} = A_{ji}$

We simply swap the indices.

Transpose swaps rows and columns



MATRIX TRANSPOSE: A^T

MATH 314 and 321 : Linear Algebra

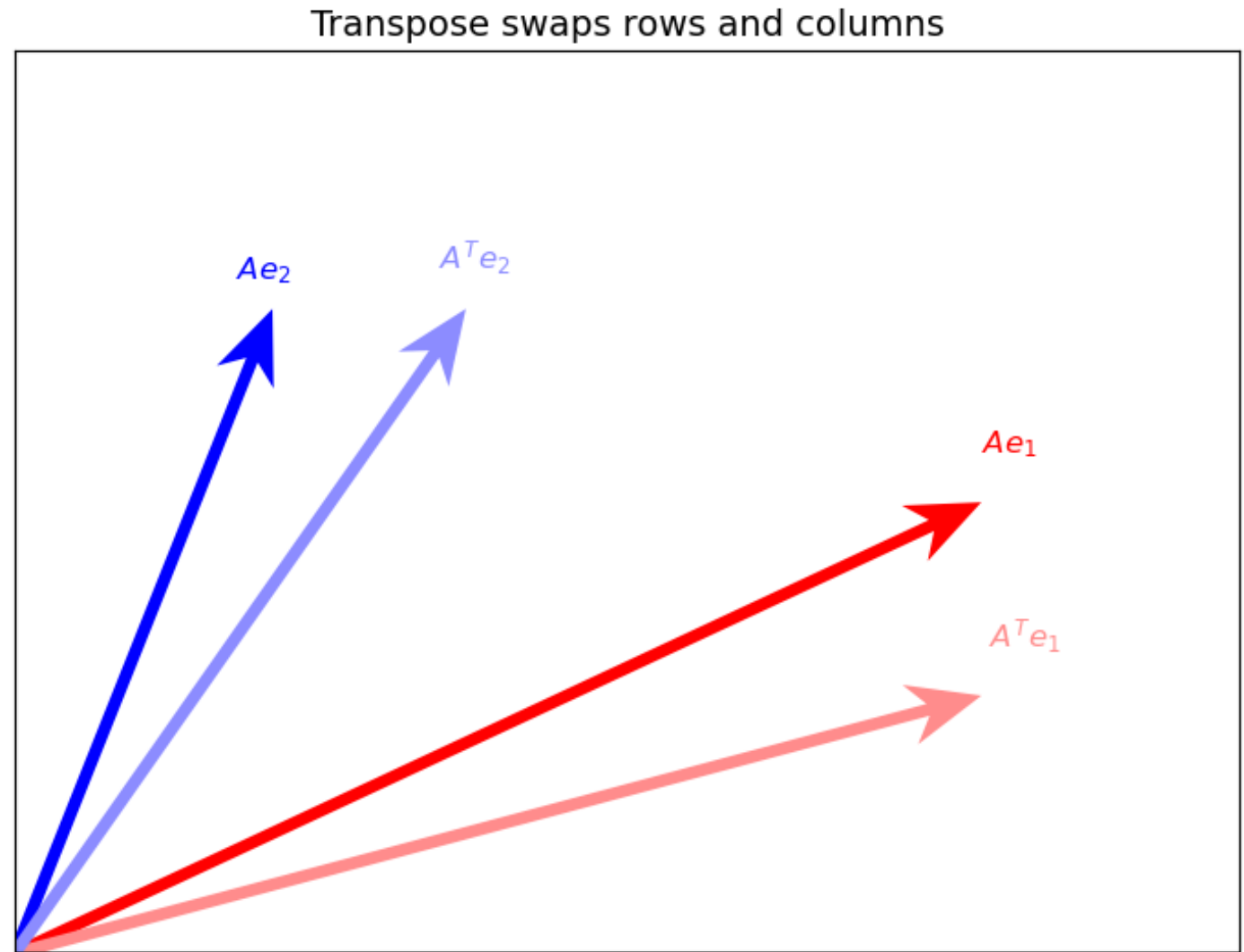
Solid arrows show where matrix A sends the basis vectors.

These are the columns of A .

Lighter arrows show where A^T sends the basis vectors.

These are the rows of A , now acting as columns.

Transpose swaps rows and columns.



ADJOINT OF A MATRIX IN REAL VECTOR SPACE

Definition:

For a linear map A on $\mathbb{R}^n$,

$$\langle Au, v \rangle = \langle u, A^\dagger v \rangle$$

In real spaces, the inner product is

$$\langle u, v \rangle = u^T v$$

So the adjoint satisfies:

$$A^\dagger = A^T$$

In real vector spaces, the adjoint is just the transpose.

The adjoint is not defined as the transpose.

It is defined by the inner product symmetry.

In real spaces, that symmetry forces the adjoint to be the transpose.

ADJOINT OF A MATRIX IN REAL VECTOR SPACE

The projection of Au onto v has exactly the same scalar value as the projection of $A^T v$ onto u

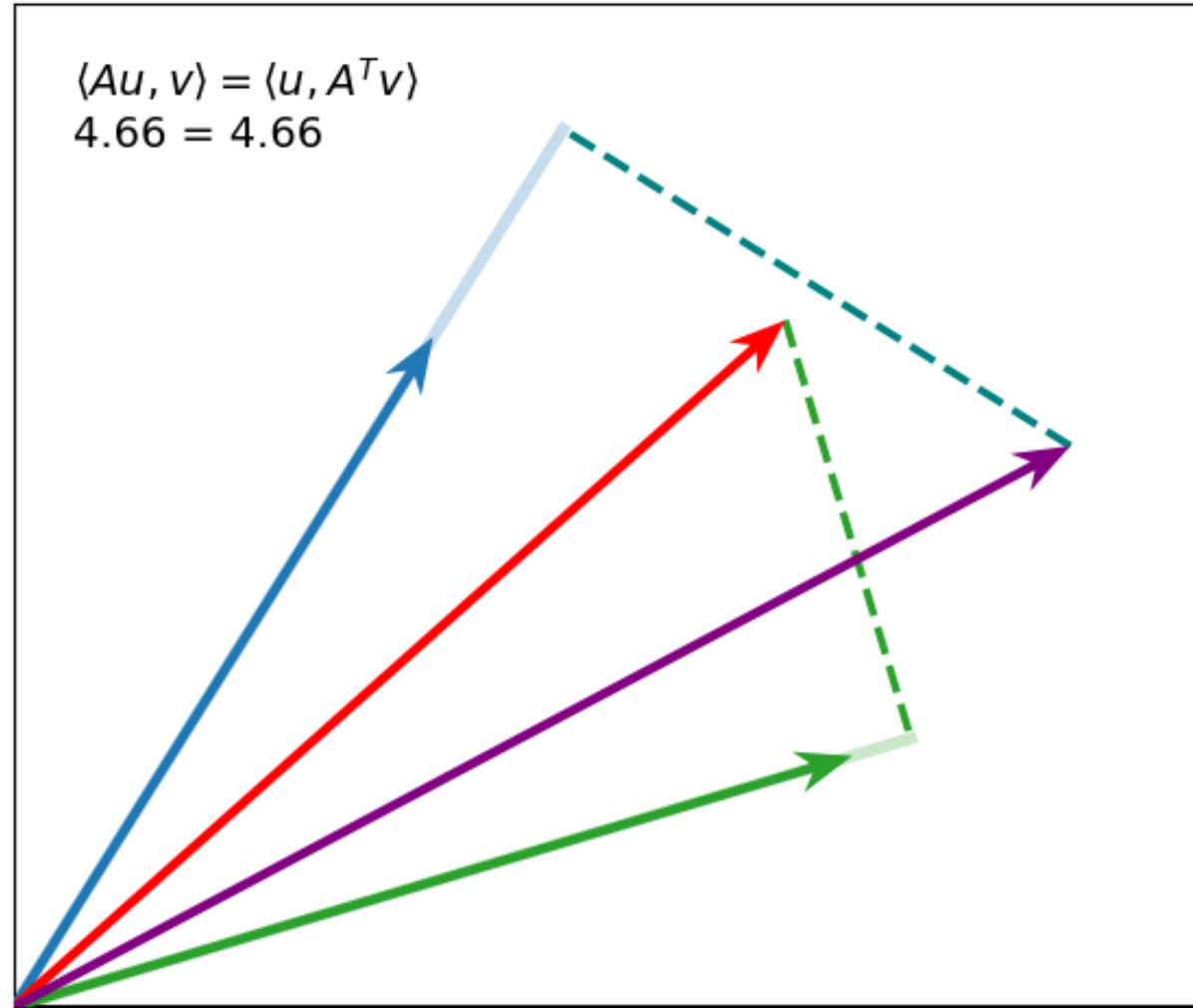
So the linear transformation can “move across” the dot product if you transpose it.

That is the adjoint definition:

$$\langle Au, v \rangle = \langle u, A^T v \rangle$$

The blue and green arrows are the original vectors u and v . The red arrow is Au , and the purple arrow is $A^T v$.

Adjoint in $\mathbb{R}^2$ (Transpose)



ADJOINT OF A MATRIX IN REAL VECTOR SPACE

The blue arrow is the vector u . The green arrow is the vector v . The red arrow is the transformed vector Au . The purple arrow is the transformed vector $A^T v$.

The dashed green segment shows the projection of the red vector Au onto the direction of the green vector v . That projection length equals $\langle Au, v \rangle$.

The dashed teal segment shows the projection of the purple vector $A^T v$ onto the direction of the blue vector u . That projection length equals $\langle u, A^T v \rangle$.

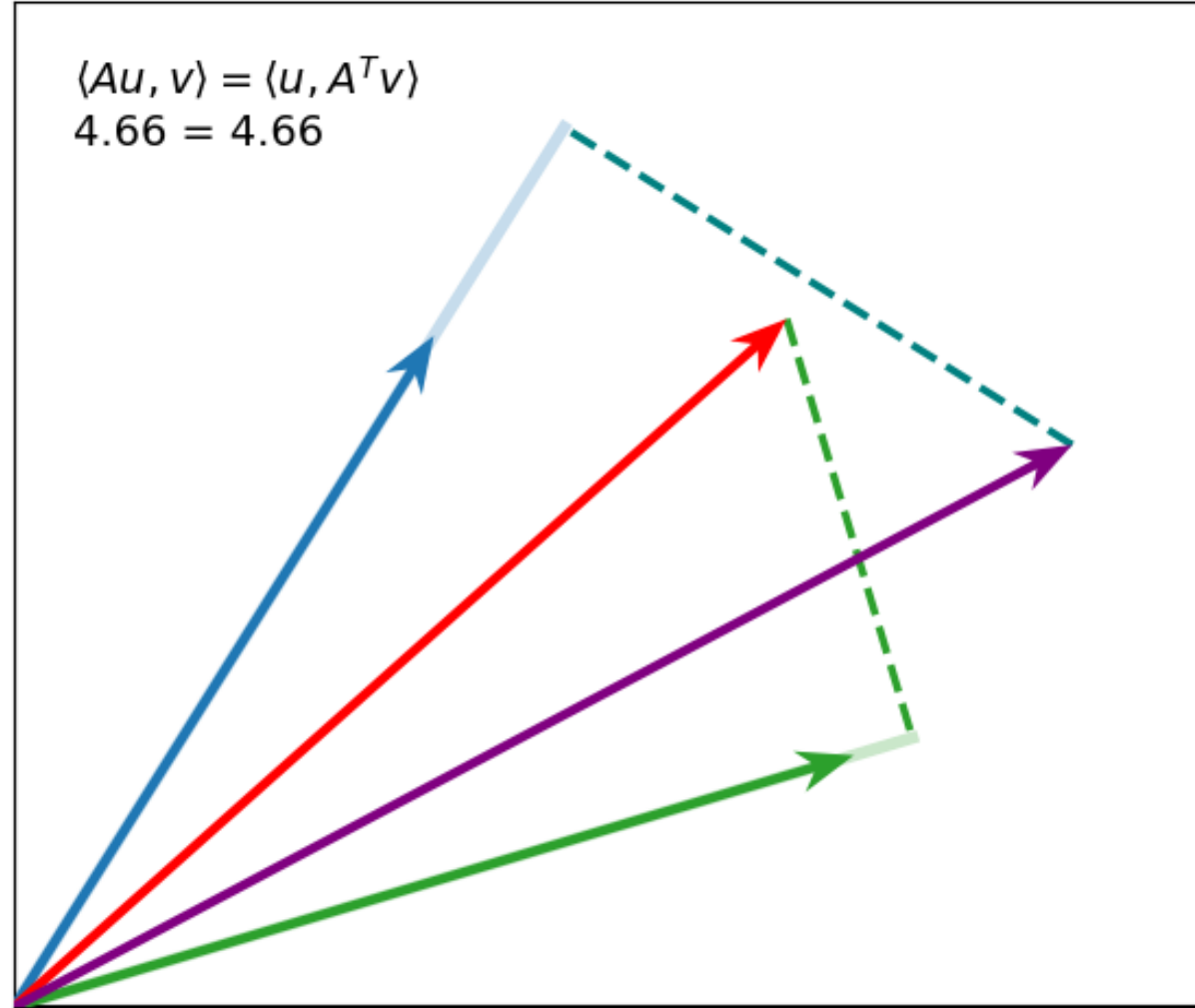
These two projection lengths are equal.

That equality is the defining property of the adjoint:

$$\langle Au, v \rangle = \langle u, A^T v \rangle$$

In a real inner product space, the adjoint of a matrix is its transpose.

Adjoint in $\mathbb{R}^2$ (Transpose)



ADJOINT OF A MATRIX IN COMPLEX VECTOR SPACE

Definition

For a linear map A , the **adjoint** $A^\dagger$ is defined by:

$$\langle Au, v \rangle = \langle u, A^\dagger v \rangle$$

for all vectors u, v .

The adjoint is the operator that:

“moves A from one side of the inner product to the other.”

It is defined relative to the inner product.

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

That is:

- Take the transpose
- Take complex conjugates

Conjugate transpose.

ADJOINT OF A MATRIX IN COMPLEX VECTOR SPACE

This is not just matrix manipulation.

The adjoint is defined by the geometry of the inner product.

It exists because we are in a Hilbert space.

The adjoint lets us define:

Hermitian matrices

Unitary matrices

Length-preserving maps

Which are the operators allowed in quantum mechanics.

LENGTH-PRESERVING TRANSFORMATIONS

A transformation preserves length if:

$$||Au|| = ||u|| \quad \text{for all } u.$$

Squaring both sides:

$$\langle Au, Au \rangle = \langle u, u \rangle.$$

Definition

A complex matrix satisfying

$$A^\dagger A = I$$

is called **unitary**.

Using the adjoint identity:

$$\langle Au, Au \rangle = \langle u, A^\dagger Au \rangle.$$

For this to equal $\langle u, u \rangle$ for all u , we must have:

$$A^\dagger A = I.$$

Geometric Meaning in $\mathbb{C}^n$

Unitary transformations preserve:

- Length
- Angles
- Inner products
- Orthogonality

They are the complex analogue of rotations.

LENGTH-PRESERVING TRANSFORMATIONS

Geometry of Normalized Vectors in $\mathbb{C}^2$

Take a vector

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with

$$|\alpha|^2 + |\beta|^2 = 1.$$

Write

$$\alpha = a + ib, \quad \beta = c + id.$$

Then:

$$a^2 + b^2 + c^2 + d^2 = 1.$$

This is the equation of a sphere in $\mathbb{R}^4$.

So:

The set of unit-length vectors in $\mathbb{C}^2$
forms a 3-sphere in $\mathbb{R}^4$.

LENGTH-PRESERVING TRANSFORMATIONS

A Concrete 2×2 Unitary Example

$$U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

Check:

$$U^\dagger = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

Need to revisit this slide.

Trying to introduce unitary maps...

So:

$$U^\dagger U = I.$$

This transformation:

- Multiplies every component by the same complex phase
- Preserves length
- Preserves inner products

It rotates each complex coordinate in its own complex plane.

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

Take a vector

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

With $|\alpha|^2 + |\beta|^2 = 1$.

Write

$$\alpha = a + ib, \quad \beta = c + id.$$

Then:

$$a^2 + b^2 + c^2 + d^2 = 1.$$

This is the equation of a sphere in $\mathbb{R}^4$.

So: The set of unit-length vectors in $\mathbb{C}^2$ forms a 3-sphere in $\mathbb{R}^4$.

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

What Unitary Transformations Do

Since unitary matrices preserve inner products,

they preserve:

“In $\mathbb{R}^2$, orthogonal matrices rotate the circle.
In $\mathbb{C}^2$, unitary matrices rotate a 3-sphere.”

$$|\alpha|^2 + |\beta|^2.$$

So they move points around on that 3-sphere.

They cannot:

- Change radius
- Stretch
- Shear

They can only rotate within that higher-dimensional sphere.

What is a Unit Sphere?

In ordinary 3-dimensional space, the unit sphere is

$$x^2 + y^2 + z^2 = 1.$$

It is the set of points at distance 1 from the origin.

That object is called a **2-sphere**, written S^2 .

Why 2? Because it is two-dimensional as a surface, even though it lives in $\mathbb{R}^3$.

The **n-sphere** S^n is:

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

It is the set of unit-length vectors in $\mathbb{R}^{n+1}$.

A **3-sphere**, written S^3 , is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$$

It lives in $\mathbb{R}^4$.

It is the set of points at distance 1 from the origin in four-dimensional space.

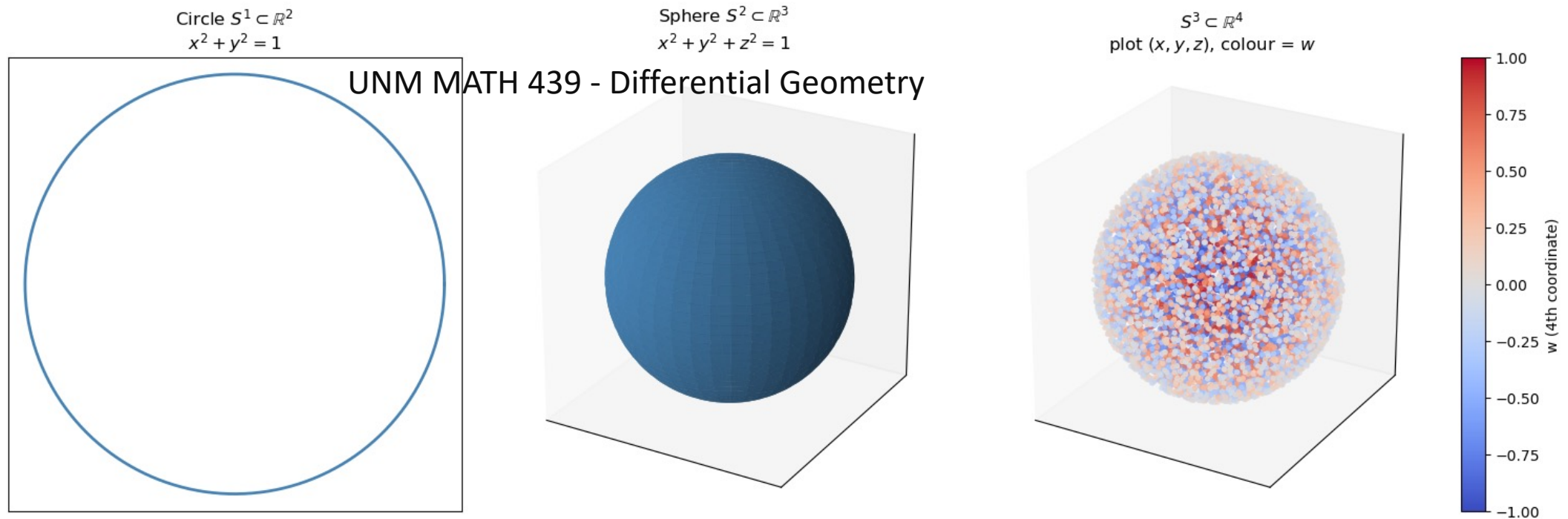
It is called a “3-sphere” because:

- Its surface is 3D, Locally, it looks like ordinary $\mathbb{R}^3$. It lives inside $\mathbb{R}^4$.

HIGHER DIMENSIONAL SPHERES

MATH 439 - Differential Geometry

Object	Equation	Lives in	Surface dimension
Circle	$x^2 + y^2 = 1$	$\mathbb{R}^2$	1
Sphere	$x^2 + y^2 + z^2 = 1$	$\mathbb{R}^3$	2
3-sphere	$x^2 + y^2 + z^2 + w^2 = 1$	$\mathbb{R}^4$	3



A circle is a 1-sphere (S^1) whose slices, obtained by fixing one coordinate in the equation above, are two points (can you see why its 2 points algebraically?); a 2-sphere (S^2) is a sphere whose slices are circles; and a 3-sphere (S^3) is a sphere whose slices are 2-spheres.

Why all this matters:

Pretty soon I will say that quantum states live on a unit sphere in complex space (S^3)

And when I say “pure quantum gates are unitary maps” you will understand that they move states around on the surface of S^3 .

You also know that this is S^3 because it's $\mathbb{C}^2$ which is $\mathbb{R}^4$.



A circle is a 1-sphere (S^1) whose slices, obtained by fixing one coordinate in the equation above, are two points (can you see why its 2 points algebraically?); a 2-sphere (S^2) is a sphere whose slices are circles; and a 3-sphere (S^3) is a sphere whose slices are 2-spheres.

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

Take a vector in $\mathbb{C}^2$:

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Write:

$$\alpha = a + ib, \quad \beta = c + id.$$

Then the norm condition

$$|\alpha|^2 + |\beta|^2 = 1$$

becomes

$$a^2 + b^2 + c^2 + d^2 = 1.$$

That is exactly the equation of a 3-sphere in $\mathbb{R}^4$. Unit-length vectors in $\mathbb{C}^2$ form S^3 .

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

Start with a vector in $\mathbb{C}^2$:

$$v = (\alpha, \beta), \quad |\alpha|^2 + |\beta|^2 = 1.$$

As we just discussed, this condition defines a 3-sphere:

$$S^3 \subset \mathbb{R}^4.$$

So unit vectors in $\mathbb{C}^2$ form S^3 .

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

Now here is the key step.

If we multiply the vector by a complex number of unit magnitude,

$$v \mapsto e^{i\phi} v,$$

its length does not change:

$$|e^{i\phi} v| = |v|$$

Geometrically, this operation moves us along a circle inside S^3 .

Each point in S^3 lies on such a circle.

GEOMETRY OF NORMALIZED VECTORS IN $\mathbb{C}^2$

If we now identify all vectors that differ only by this phase factor,
 $v \sim e^{i\phi} v$,

we are collapsing each of those circles to a single point.

Mathematically, this quotient space is

$$S^3/S^1.$$

And a theorem from geometry says:

$$S^3/S^1 \cong S^2.$$

So when you remove the redundant phase degree of freedom from unit vectors in $\mathbb{C}^2$, what remains is a 2-sphere.

QUBITS



Bloch Sphere (unit 2-sphere)

That 2-sphere is what later gets called the Bloch sphere.

So the reduction is:

$$\text{unit vectors in } \mathbb{C}^2 \longrightarrow S^3 \longrightarrow \frac{S^3}{S^1} \cong S^2.$$

In quantum computing instead of having bits that can be 0 or 1 like we do in classical computing, we have qubits whose state is a point on this 2-sphere.

Unitary maps are rotations of this sphere and define the operations we can perform on qubits.

