



COMPLEX VECTORS AND INNER PRODUCTS

Lecture 4

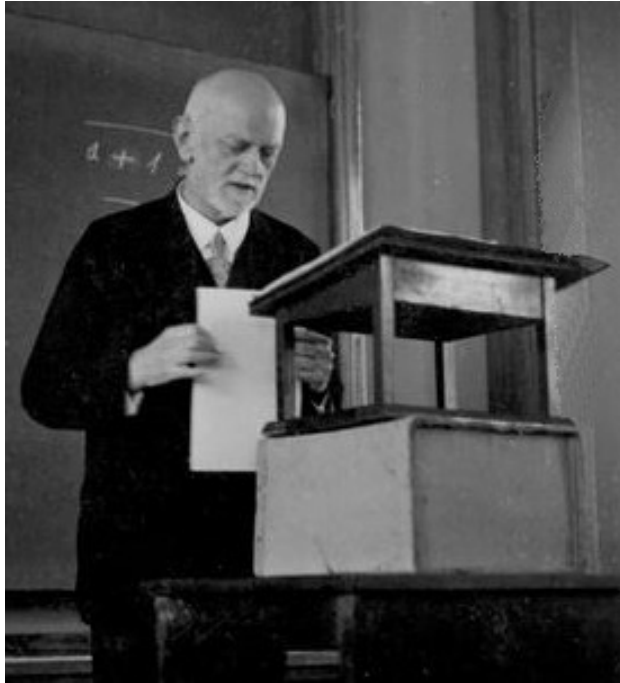
LAST TIME: EULER'S EQUATION

- We already showed that:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

We can use that formula to rotate and scale complex vectors. Which is about all you can do to a vector in a 2D plane.

- We also previewed that in quantum computation we will require that the complex vectors have modulus 1 so they inscribe a unit circle on the complex plane.
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David Hilbert in 1932

Grundlagen der Geometrie (Foundations of Geometry), 1899

University of Göttingen

“We must know. We will know.”

(*Wir müssen wissen. Wir werden wissen.*)

“Mathematics in Göttingen? There is none anymore.” reaction to the Nazi expulsion of Jewish faculty in 1933.

THIS TIME: COMPLEX VECTORS AND HILBERT SPACES

- Quantum computing is built on a mathematical structure called a **Hilbert space**.
- In this lecture, we begin developing the geometry of complex vector spaces that will allow us to understand what a Hilbert space is and why quantum states live there.
- Hilbert’s program sought a complete and consistent formal foundation for mathematics.
- As you will see in your CS theory courses Kurt Godel and Alan Turing showed Hilbert’s dream was impossible.

VECTOR SPACE GEOMETRY

Real Numbers

One coordinate.
One degree of freedom.
A number line.

Pairs of Real Numbers

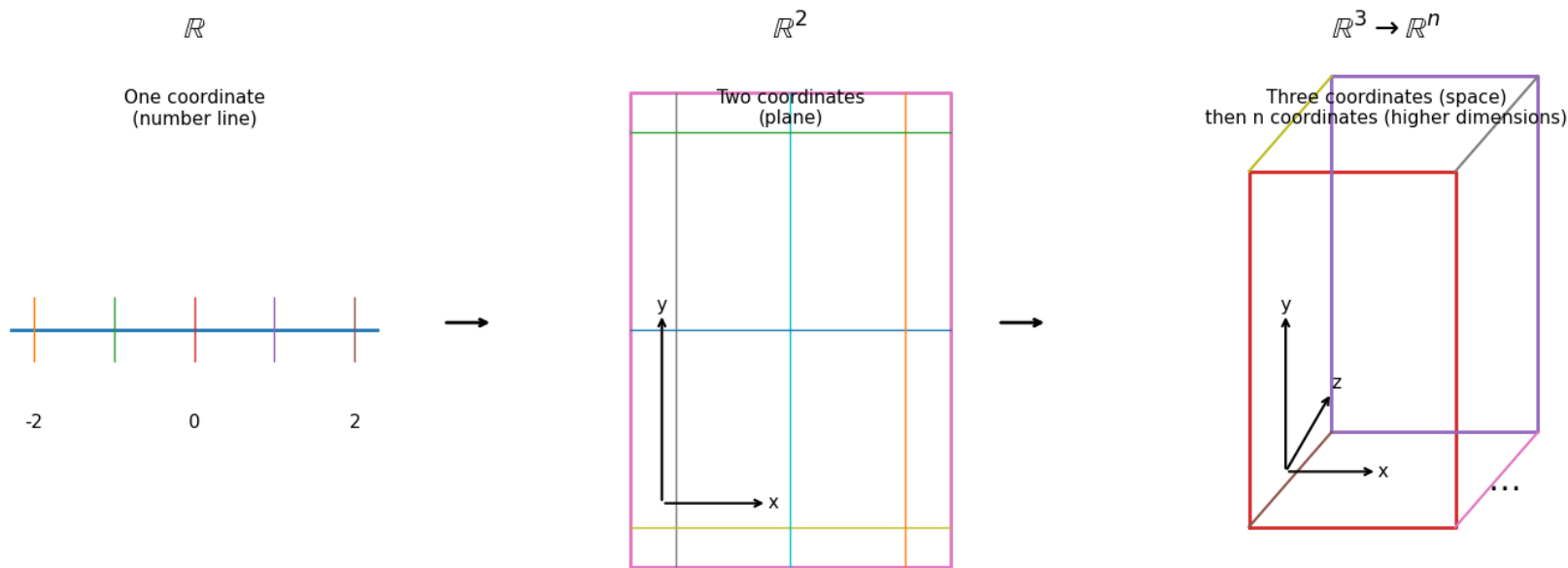
$\mathbb{R}^2 = (xy)$
Two independent coordinates.
Two degrees of freedom.
A plane.

Triples of Real Numbers

$$\mathbb{R}^3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Three independent coordinates.
Three degrees of freedom.
Ordinary space.

From Lines to Planes to Higher Dimensional Spaces



COMPLEX NUMBER ARE A SINGLE VALUE

- Up to now, we have viewed a complex number as

$a + ib$ or equivalently as the point (a, b) in the complex plane.

Now we change perspective.

- We will treat a complex number as a **single scalar value**, just as a real number is a scalar.
- The geometry we have described (modulus and argument*) is the geometry of the scalar field $\mathbb{C}$.
- Once we do this, we can build higher-dimensional spaces:
 - A single complex number lives in $\mathbb{C}$
 - A pair of complex numbers lives in $\mathbb{C}^2$
 - In general, n complex numbers form $\mathbb{C}^n$
 - Each point in $\mathbb{C}^n$ is a vector whose components are complex scalars.

*As with the rest of language there are many words for the same thing: Modulus: magnitude, length, size; Argument: angle, phase, rotation (rotation is an operation though not a property.)

COMPLEX NUMBER ARE A SINGLE VALUE

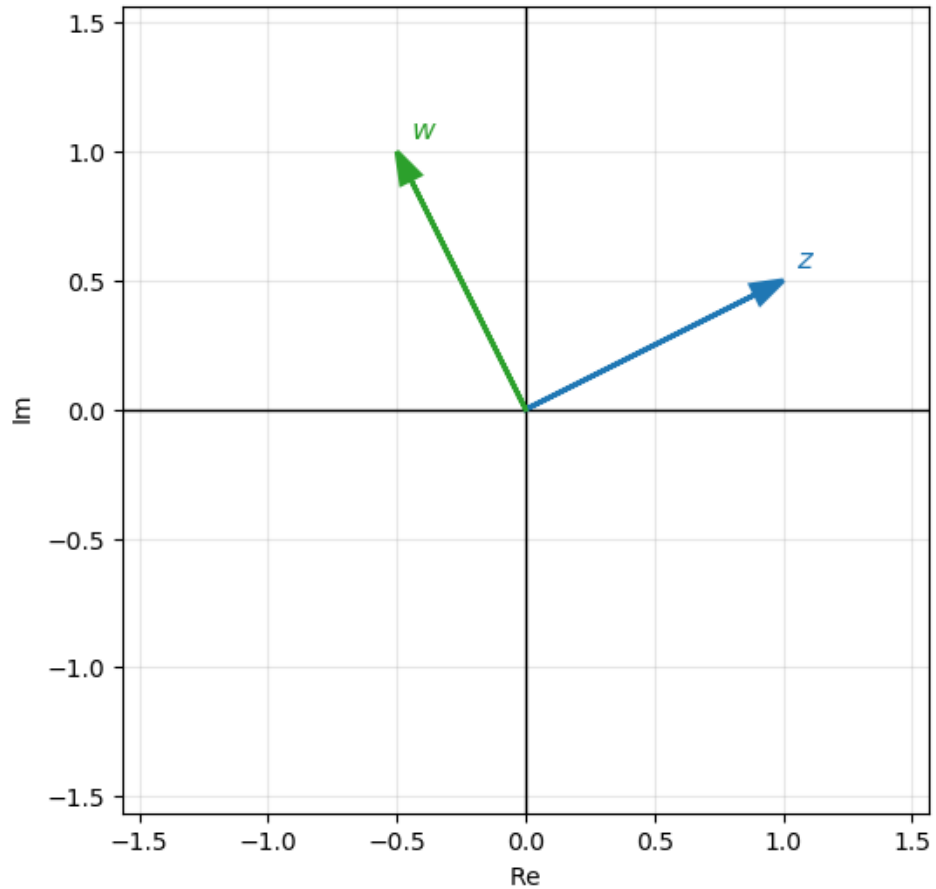
- The complex plane is $\mathbb{C}$ viewed as $\mathbb{R}^2$.
- The “plane of complex numbers” in quantum computing is often $\mathbb{C}^2$ — which is 4-dimensional over $\mathbb{R}$.

Since we do this, we can build higher dimensional spaces:

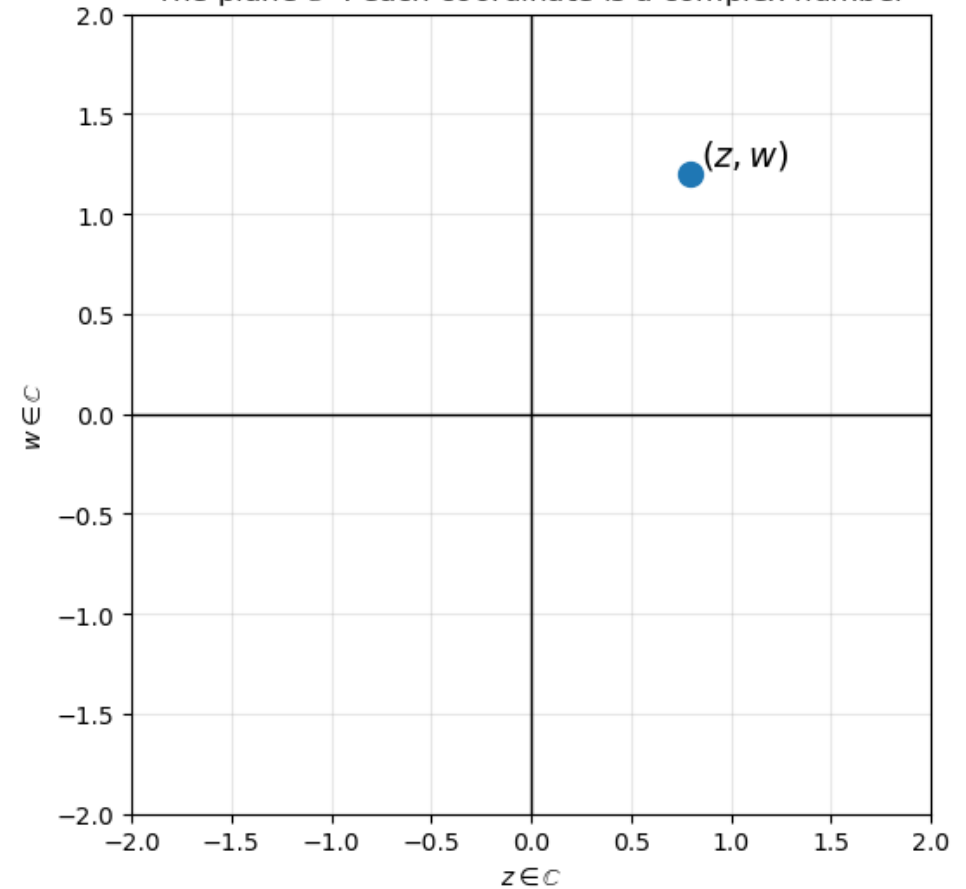
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THE 2D COMPLEX VECTOR SPACE: $\mathbb{C}^2$

Complex arithmetic as geometry



The plane $\mathbb{C}^2$: each coordinate is a complex number



Complex Plane



WHY WE CARE...

- In quantum computing we will represent qubits with two complex numbers. That means they live in $\mathbb{C}^2$.

Preview: A qubit $|\psi\rangle$ is two complex numbers:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad a, b \in \mathbb{C}.$$

- Multiplication of complex numbers adds angles and multiplies lengths. In quantum computing, those angles become *phase*.
 - A pair of complex numbers gives us *superposition* and *interference*, which are two of the central concepts in quantum computing.
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MEASUREMENT IN $\mathbb{R}^2$ GEOMETRY

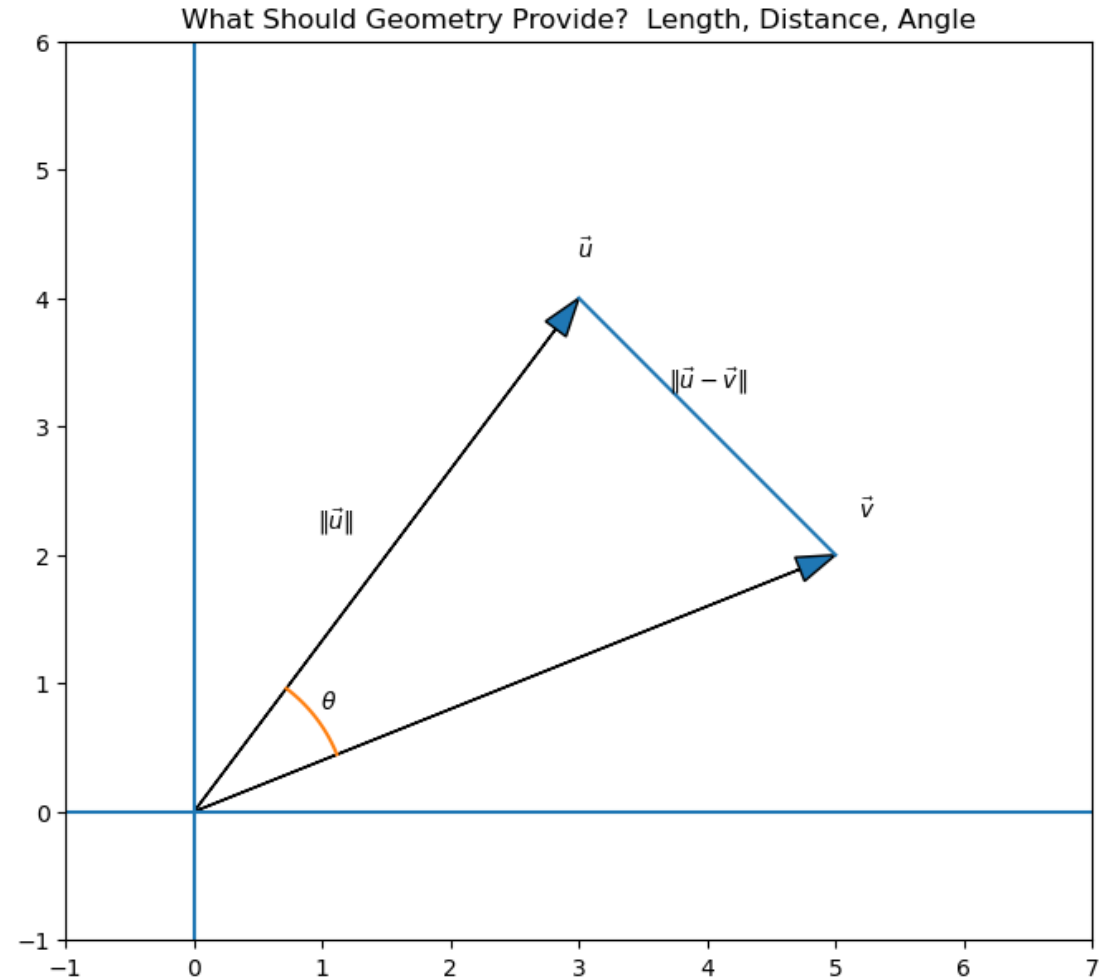
What Should Geometry Provide?

If we are describing points in space, what do we want to measure?

At minimum:

1. **Length** of a vector: $|\vec{v}|$
2. **Distance** between two points: $|\vec{u} - \vec{v}|$
3. **Angle** between two directions: θ

These are the fundamental geometric quantities.



LENGTH OF A VECTOR IN $\mathbb{R}^2$

Let $\vec{v} = (a, b)$

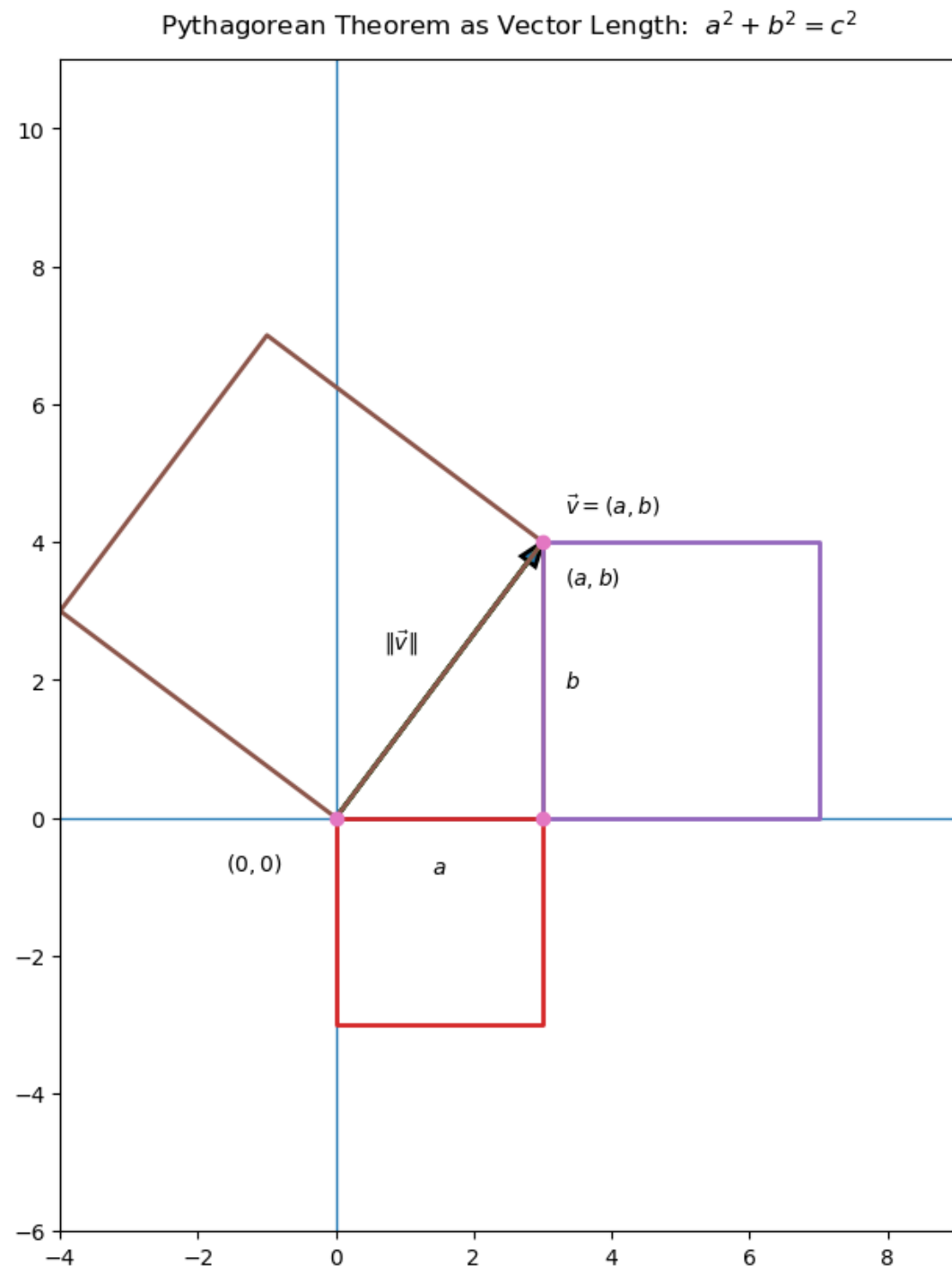
By the Pythagorean theorem, $\|\vec{v}\|^2 = a^2 + b^2$

So the **length** of $\vec{v}$ is $\|\vec{v}\| = \sqrt{a^2 + b^2}$.

Then generalise to higher dimensions: $\vec{v} = (v_1, v_2, \dots, v_n)$

Then $\|\vec{v}\|^2 = v_1^2 + v_2^2 + \dots + v_n^2$

And $\|\vec{v}\| = \sqrt{\sum_{i=1}^n v_i^2}$.



DISTANCE BETWEEN TWO POINTS IN $\mathbb{R}^2$

Distance Between Points: $\text{distance}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$.

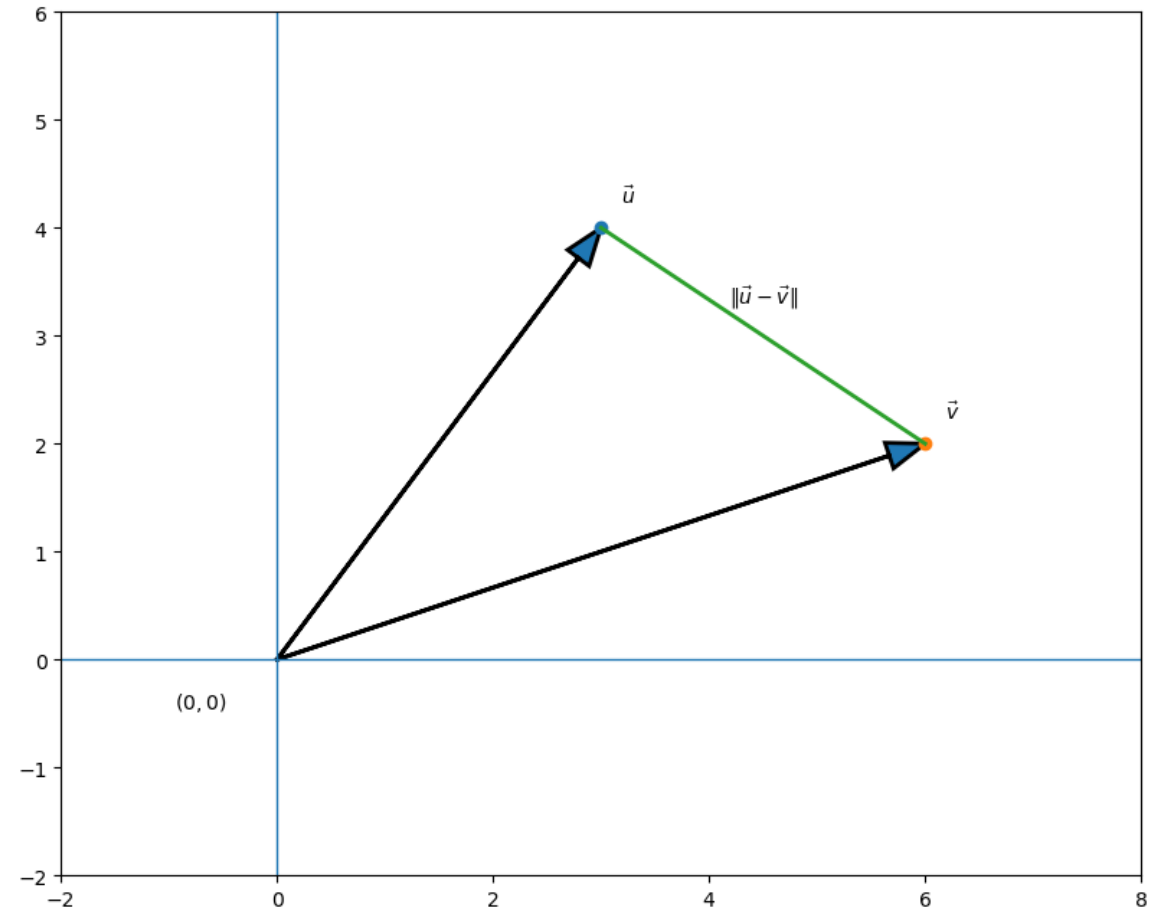
We can treat their tips as points in the plane.

The **distance** between those points is the length of their difference:

$$\text{distance}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|.$$

If $\vec{u} - \vec{v} = (\Delta x, \Delta y)$, then by Pythagoras:

$$\|\vec{u} - \vec{v}\| = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$



ANGLES BETWEEN $\mathbb{R}^2$ VECTORS

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^2$.

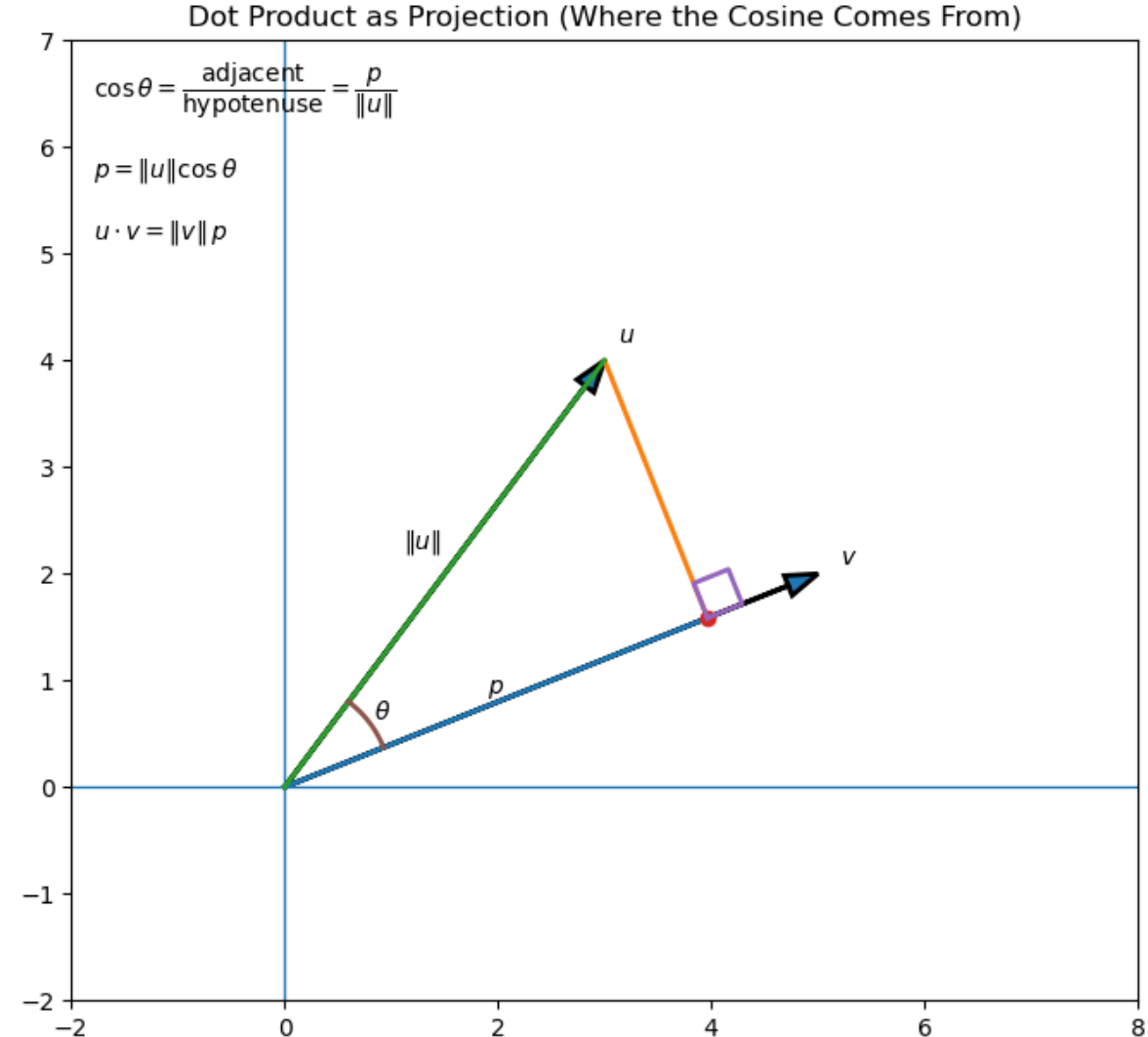
The **angle** θ between them measures how much their directions align.

We define θ using:

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{p}{\|u\|} \quad \Rightarrow \quad p = \|u\| \cos \theta.$$

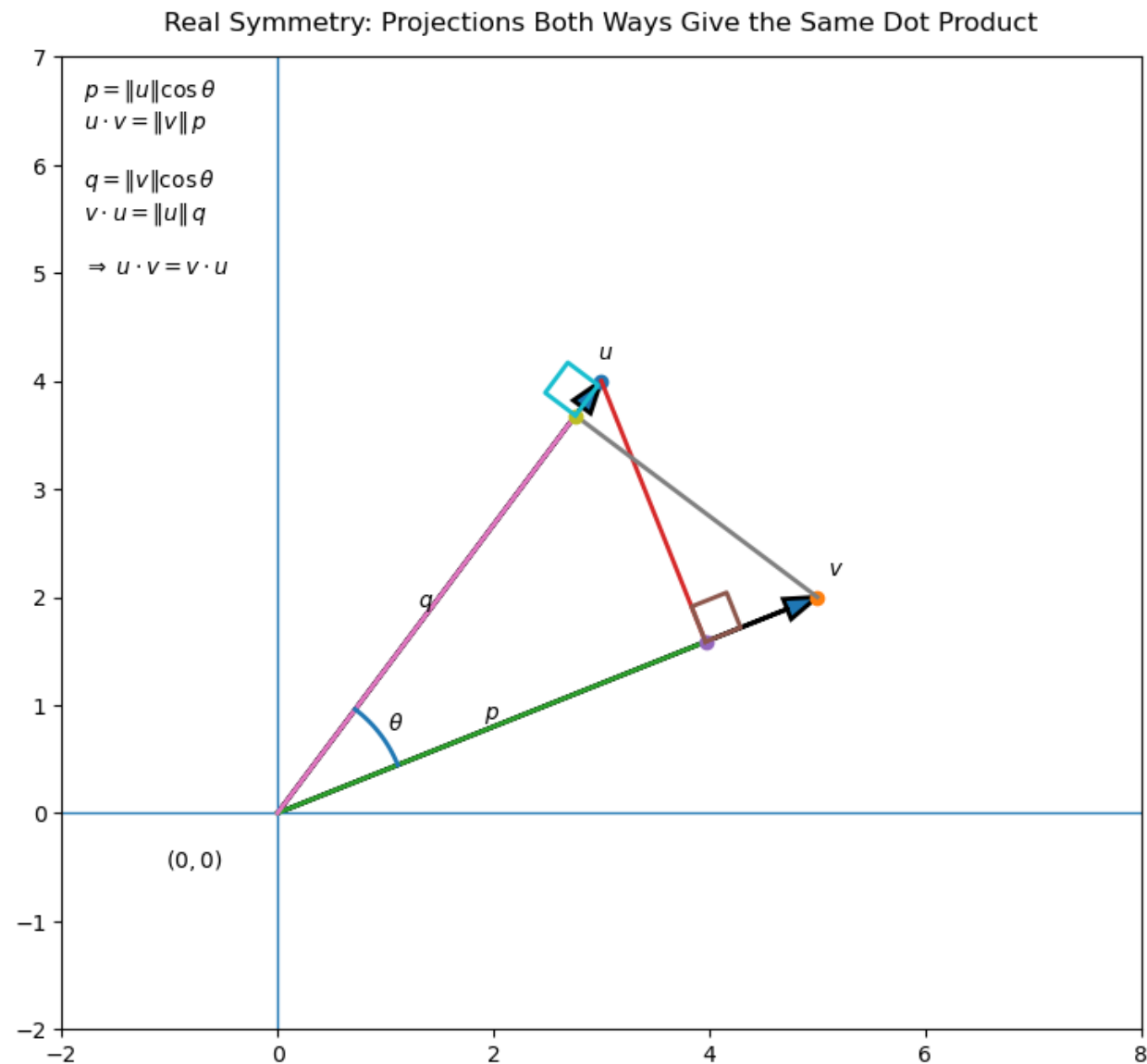
$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|, \|\vec{v}\|}.$$

This is equivalent to:



SYMMETRY OF ANGLES

Symmetry ensures that geometric measurement in $\mathbb{R}^2$ is mutual and order-independent.



MEASUREMENT IN $\mathbb{C}^2$ GEOMETRY

- We want all those in our complex space so we can reason about how complex vectors relate to each other.

LENGTH OF A VECTOR IN $\mathbb{C}^2$ (WRONG)

Let

$$\vec{z} = (z_1, z_2), \quad z_1, z_2 \in \mathbb{C}.$$

If we copy the real formula, we might try*:

$$|\vec{v}|^2 = v_1^2 + v_2^2$$

But complex numbers have imaginary parts.

Example:

Let

$$\vec{z} = (1, i).$$

Then

$$z_1^2 + z_2^2 = 1^2 + i^2 = 1 - 1 = 0.$$

So a nonzero vector would have zero length.

This cannot be correct.

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COMPLEX CONJUGATION

Definition

For a complex number

$$z = a + bi$$

the **complex conjugate** is

$$\bar{z} = a - bi$$

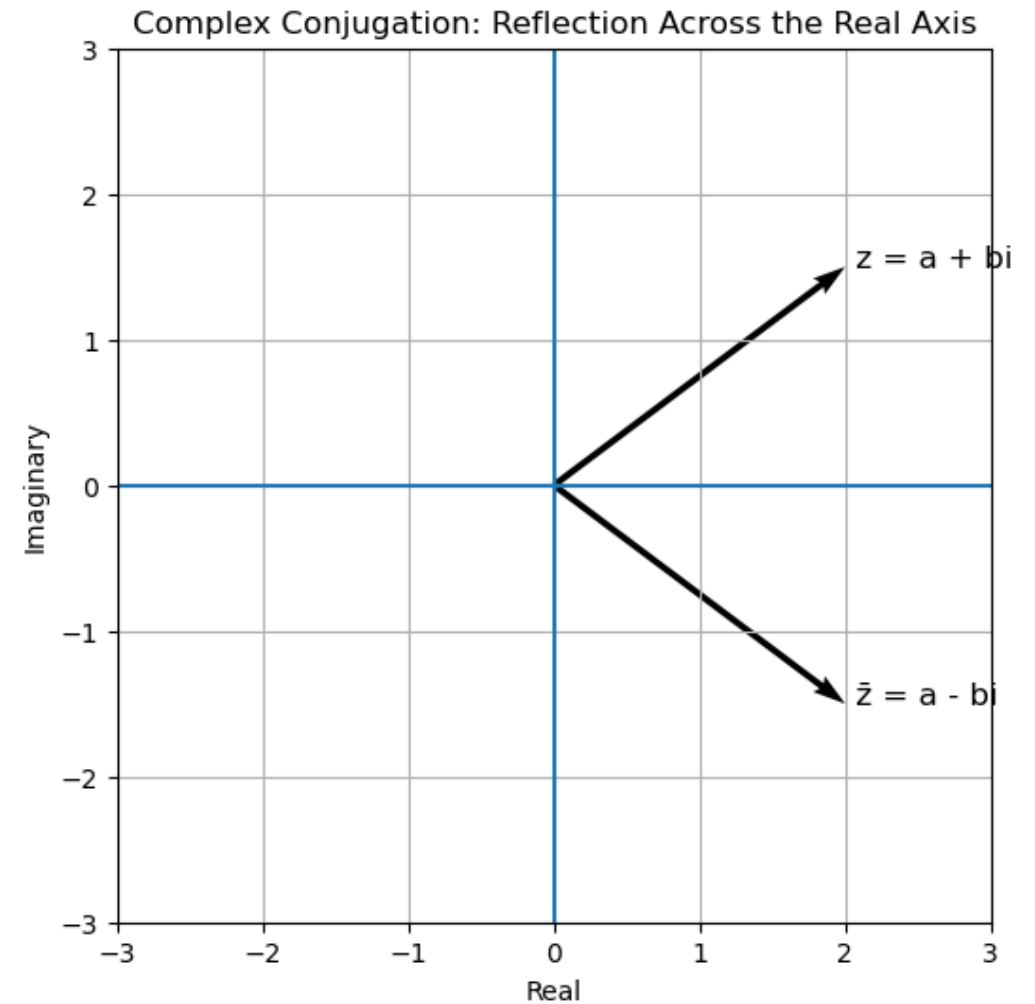
Geometric interpretation

- Conjugation reflects a point across the **real axis**
- The real part stays the same
- The imaginary part changes sign

$$z\bar{z} = a^2 + b^2 = |z|^2$$

Converts complex values into a **real, positive length**

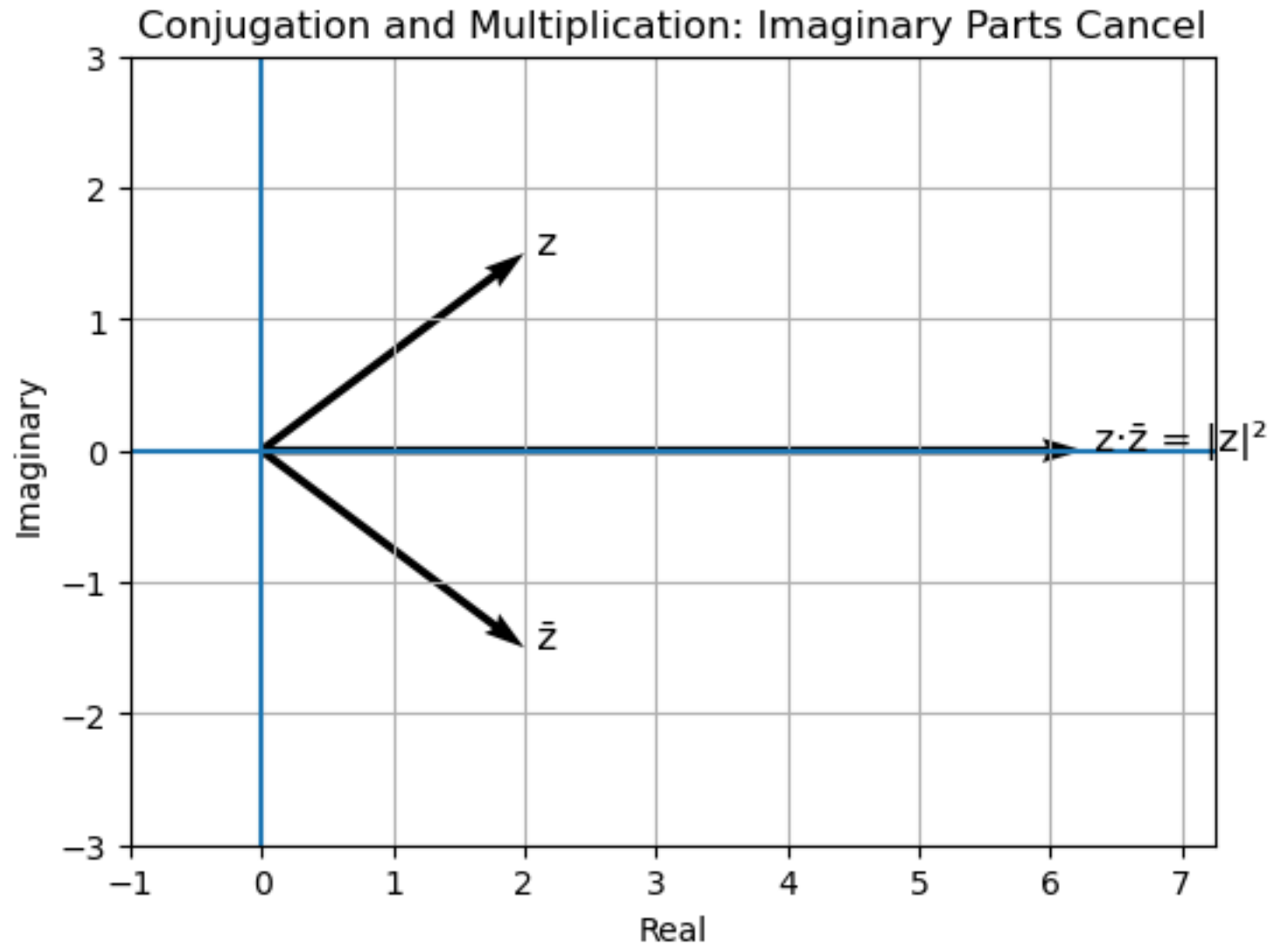
This is what makes geometry in complex space work



COMPLEX CONJUGATION

These two vectors are mirror images.

When we multiply them, the imaginary parts cancel, and the result lands entirely on the real axis, giving us the squared length.



LENGTH OF A VECTOR IN $\mathbb{C}^2$ (FIXED)

For a complex number:

$$|z|^2 = z\bar{z}.$$

So instead define:

$$|\vec{v}|^2 = z_1\bar{z}_1 + z_2\bar{z}_2.$$

Now length is always real and nonnegative.

Example:

$$\begin{aligned}\vec{z} &= (1, i) \\ |\vec{z}|^2 &= 1 \cdot 1 + i(-i) = 1 + 1 = 2.\end{aligned}$$

Now the result makes sense.

This fix is going to propagate to the Distance and angle functions since they also contain information about length.

ANGLES BETWEEN $\mathbb{C}^2$ VECTORS

Real case:

$$u \cdot v = a_1 a_2 + b_1 b_2.$$

Complex case must preserve length.

So define:

$$\langle u, v \rangle = u_1 \overline{v_1} + u_2 \overline{v_2}.$$

Then

$$\langle v, v \rangle = |v_1|^2 + |v_2|^2 = \|v\|^2.$$

ANGLES BETWEEN $\mathbb{C}^2$ VECTORS

Real Case: $\mathbb{R}^2$

$$u \cdot v = v \cdot u$$

$$v \cdot v = a_1^2 + a_2^2 + \cdots + a_n^2$$

Everything is symmetric.

Length is:

$$|v|^2 = v \cdot v.$$

In real space: symmetry.

In complex space: conjugate symmetry.

Complex Case: $\mathbb{C}^n$

We must define:

$$\langle u, v \rangle = u_1 \overline{v_1} + \cdots + u_n \overline{v_n}.$$

Now symmetry changes:

$$\langle u, v \rangle = \overline{\langle v, u \rangle}.$$

Length is:

$$|v|^2 = \langle v, v \rangle = |v_1|^2 + \cdots + |v_n|^2.$$

MEASUREMENTS IN $\mathbb{C}^2$

- It is tricky to show the geometry because remember that $\mathbb{C}^2$ is actually $\mathbb{R}^4$ and I am bad at drawing 4D objects.

1) Length

In $\mathbb{R}^n$ we wrote the inner product as $u \cdot v$.

In $\mathbb{C}^n$, we use the notation $\langle u, v \rangle$ to emphasise the role of complex conjugation.

• 2) Distance

$$d(u, v) = |u - v| = \sqrt{\langle u - v, u - v \rangle}$$

• 3) Angle

$$\cos(\theta) = \frac{\operatorname{Re}\langle u, v \rangle}{\|u\|\|v\|} \langle u, v \rangle = \|u\|\|v\|e^{i\theta}$$

MEASUREMENTS IN $\mathbb{C}^N$

It is not that these formulas happen to work for $\mathbb{C}^n$.

For

$$u = (u_1, \dots, u_n), \quad v = (v_1, \dots, v_n) \in \mathbb{C}^n,$$

1) Length

$$\|v\|^2 = \langle v, v \rangle = \sum_{k=1}^n v_k \overline{v_k} = \sum_{k=1}^n |v_k|^2$$

It is that we define $\mathbb{C}^n$ to be complex spaces that follow those formulas.

2) Distance

$$d(u, v) = |u - v| = \sqrt{\langle u - v, u - v \rangle}$$

More generally they are a member of the *inner-product spaces*, because the inner-product defines them.

3) Angle

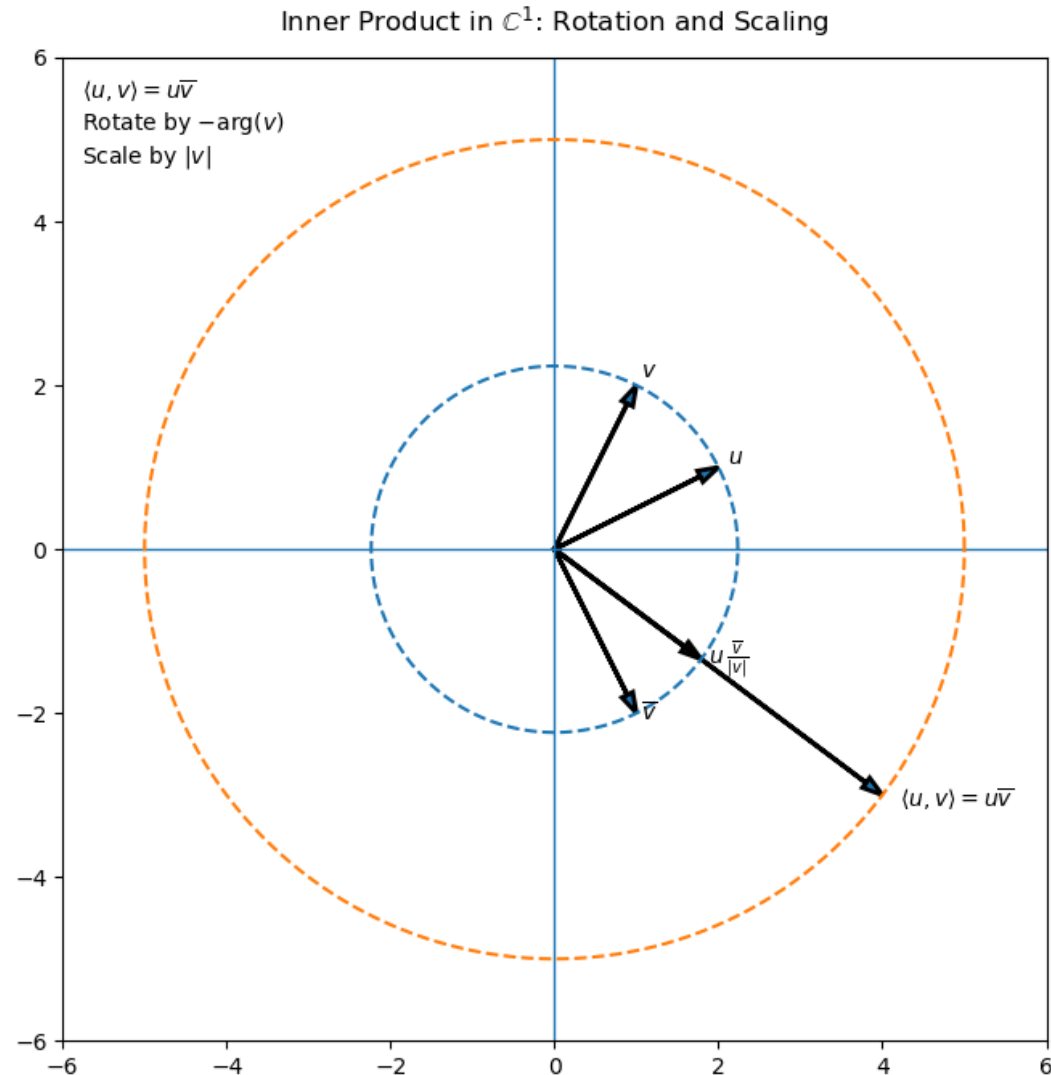
$$\langle u, v \rangle = |\langle u, v \rangle| e^{i\theta}$$

COMPLEX VECTOR SPACES

- $\langle u, v \rangle = |\langle u, v \rangle| e^{i\theta}$ The inner product is a complex number with a magnitude and a phase.
 - $|\langle u, v \rangle| \leq \|u\| \|v\|$ **Cauchy–Schwarz inequality:** The overlap between two vectors cannot exceed the product of their lengths.
 - $\cos(\theta) = \frac{\operatorname{Re} \langle u, v \rangle}{\|u\| \|v\|}$ **Angle definition:** The cosine of the angle is determined by the real part of the inner product relative to the vector lengths.
-

COMPLEX INNER PRODUCT

- To build intuition for conjugation, we first look at $\mathbb{C}^1$.
- In higher dimensions, the inner product is a sum of these one-dimensional pieces.
- The inner product of complex vectors u and v rotates u by $-\arg(v)$ and scales the result by $|v|$



HILBERT SPACES

- A Hilbert space is a complex vector space equipped with an inner product and complete with respect to the induced norm.
 - Quantum states live in complex Hilbert spaces.
 - $\mathbb{C}^n$ with this inner product is already complete.
 - Therefore $\mathbb{C}^n$ is a Hilbert space.
-

COMPLEX HILBERT SPACES ARE VECTOR SPACES

Vector spaces do NOT automatically give us:

- A notion of length
- A notion of angle
- A notion of distance
- Convergence of infinite sequences

A vector space provides algebra.
Geometry and limits require more.

A set V is a complex

Algebraic rules hold

Associativity: $(u +$

Commutativity: $u +$

Additive Identity: $v +$

Additive Inverse: $v +$

Distributivity: $\alpha(u +$

α and β are scalars.

All that guarantees

We can form all lin

COMPLEX HILBERT SPACES: INNER PRODUCTS

A complex vector space gives us algebra.

An inner product gives us geometry.

From the inner product we obtain:

$$|v| = \sqrt{\langle v, v \rangle}$$
$$\text{Angle via } \cos \theta = \frac{\operatorname{Re} \langle u, v \rangle}{|u||v|}$$

This allows us to:

- Measure length
- Measure distance
- Define orthogonality
- Decompose vectors into components

Without an inner product there is no notion of length, angle, or orthogonality.

$\operatorname{Re}(z)$ means return just the real part

HILBERT SPACES: COMPLETE

- $\forall \varepsilon > 0, ; \exists N$ such that $m, n > N \Rightarrow \|v_m - v_n\| < \varepsilon$.

Completeness just means that there are no gaps in the space.

This is pretty trivial for $\mathbb{C}^n$ and that's all we need for quantum computation.

The fullness of quantum physics also cares about completeness in $\mathbb{C}^\infty$ but that is for another class.

XENO'S PARADOX

Zeno's Paradox (Achilles)

Zeno argues:

To reach the finish line, you must first go halfway.

Then half of the remaining distance.

Then half again.

And so on forever.

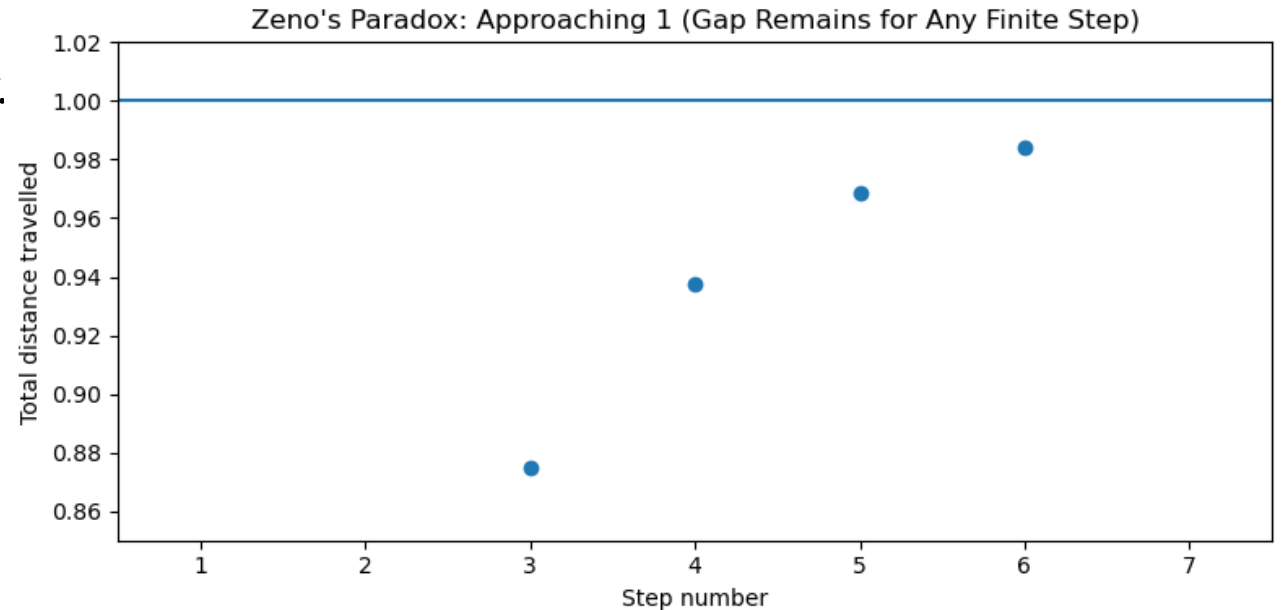
So how can motion ever finish?

Because there are infinitely many steps.

Zeno of Elea

Date: circa 450 BCE

Source: The paradoxes are known primarily from Aristotle's *Physics* (Book VI), where Aristotle discusses and critiques them.



XENO'S PARADOX AND COMPLETE SPACES

- **Aristotle** (4th century BCE) responded philosophically: infinitely many divisions of distance do not require infinite time.
 - In the 19th century, Augustus Cauchy and Karl Weierstrass made limits rigorous. They showed that an infinite sequence of shrinking steps can converge to a finite value.
 - The key mathematical property is **completeness**:
 - A complete space guarantees that if a sequence converges, its limit actually exists inside the space.
 - For the real numbers, $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, so in the limit Achilles does reach the finish line, because the real numbers contain the limit point.
-

NOT EVERY SPACE IS COMPLETE

The Rational Numbers $\mathbb{Q}$ Consider the space of fractions i.e. every element is a ratio of integers.

Define a sequence that converges on $\sqrt{2}$, i. e. $\lim_{n \rightarrow \infty} \frac{1}{2} \left(x_n + \frac{2}{x_n} \right)$

Each term is rational.

These numbers get closer and closer to $\sqrt{2}$ as n increases.

But the final destination doesn't exist in $\mathbb{Q}$.

$\sqrt{2} \notin \mathbb{Q}$.

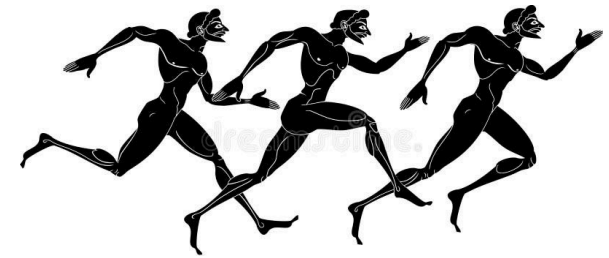
So the sequence converges in $\mathbb{R}$,

but its limit does not exist in $\mathbb{Q}$. In fact none of the irrational numbers (e, pi, etc) exist in $\mathbb{Q}$ so it is full of holes.

~~There is a sequence in the space whose terms get arbitrarily close to some point,~~
and that point ought to be the limit, but it is not in the space. So it feels like a hole.

COMPLEX HILBERT SPACES: COMPLETE

- A Hilbert space is:
- A complex vector space with an inner product that has no missing limit points.
- Completeness guarantees:
 - Infinite sums of vectors converge to a vector in the space.
 - Cauchy sequences of states have limits inside the space.
 - Approximations converge to actual physical states.
- If a sequence of quantum states converges, its limit is also a valid state.
- You are guaranteed to reach the missing finish line if you converge on it.



QUANTUM COMPUTATION IS LINEAR ALGEBRA IN A COMPLEX HILBERT SPACE.

Quantum programs manipulate vectors in $\mathbb{C}^m$.

To understand what a quantum program is doing, you must understand:

- Superposition $\rightarrow$ linear combinations
- Normalization $\rightarrow$ length using inner product
- Measurement $\rightarrow$ inner products determine probabilities
- Unitary gates $\rightarrow$ preserve inner product and length

(I haven't told you what any of the things on the left above are yet)

