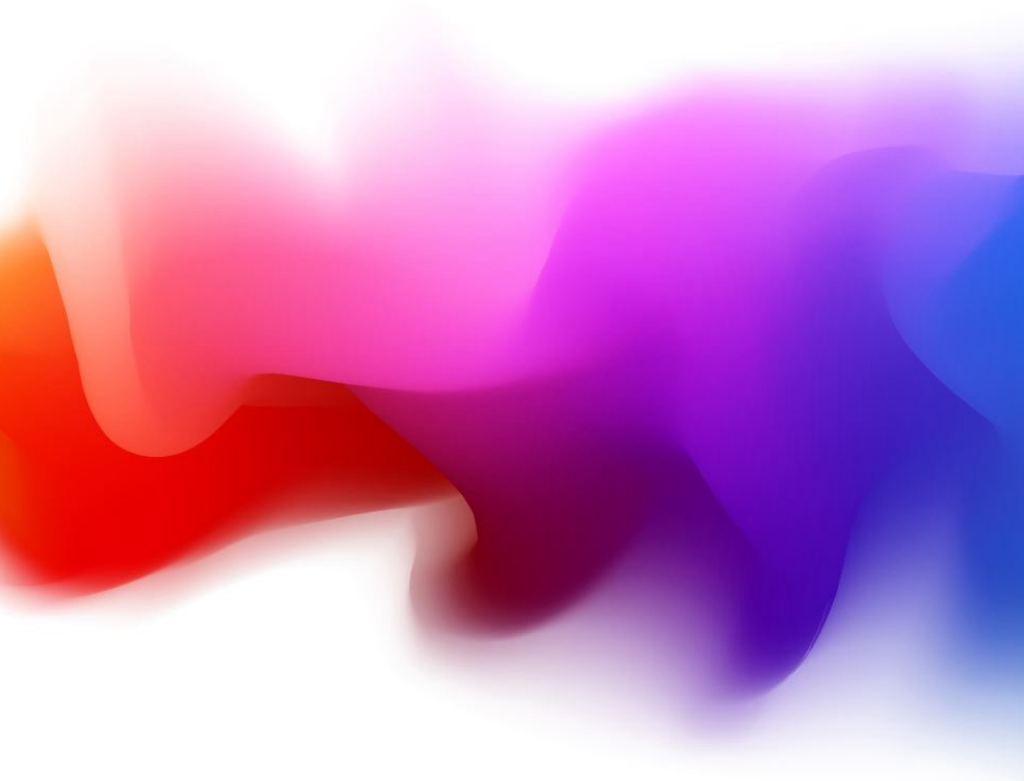




COMPLEX GEOMETRY

Lecture 3

GOALS

- By the end of this lecture, you should be able to:
- Treat complex numbers as vectors in the plane and reason geometrically about addition and multiplication.
- Use polar form to compute powers and roots, and explain why roots form regular polygons.
- Apply De Moivre's formula and connect it to the complex exponential form.
- Write Python in a Jupyter notebook to visualise complex transformations and the n th roots of unity.
- Understand Möbius transformations as “fractional linear” maps that send lines and circles to lines and circles.

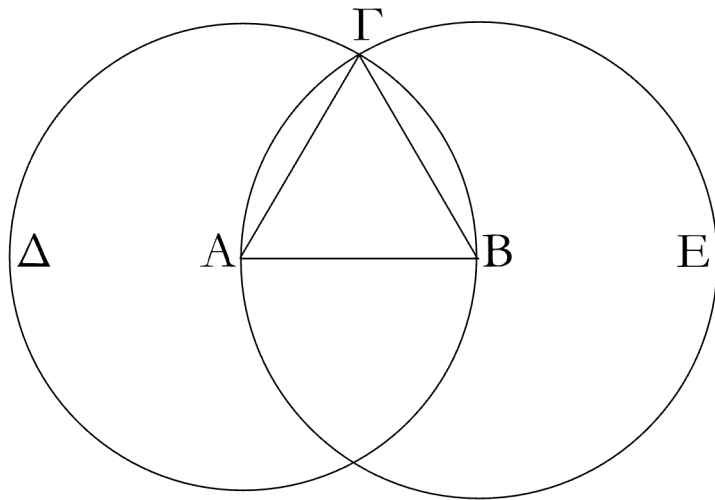
VECTORS IN THE COMPLEX PLANE

- **Complex numbers as 2D vectors**
 - Last lecture we defined complex numbers and showed how they exist in a 2d plane formed by the combination of the real and imaginary number lines.
 - Vectors are mathematical objects that let us describe relationships in the plane easily
 - Instead of always writing $a + bi$, we will shorten things to $[a, b]$ this is not just for convenience. That notation is general and allows us to apply all sorts of powerful mechanisms.
 - $\vec{v} = [a, b]$ means vector “v” with components or elements a and b
 - Vectors are ubiquitous in math and physics and so have developed a lot of different notation over the years. Physicists often put an arrow over the variable to indicate a vector while mathematicians lean towards bold face: $\vec{v}$ vs. $\mathbf{V}$.
 - J. Willard Gibbs formalised vector calculus in *Vector Analysis*, 1901
-

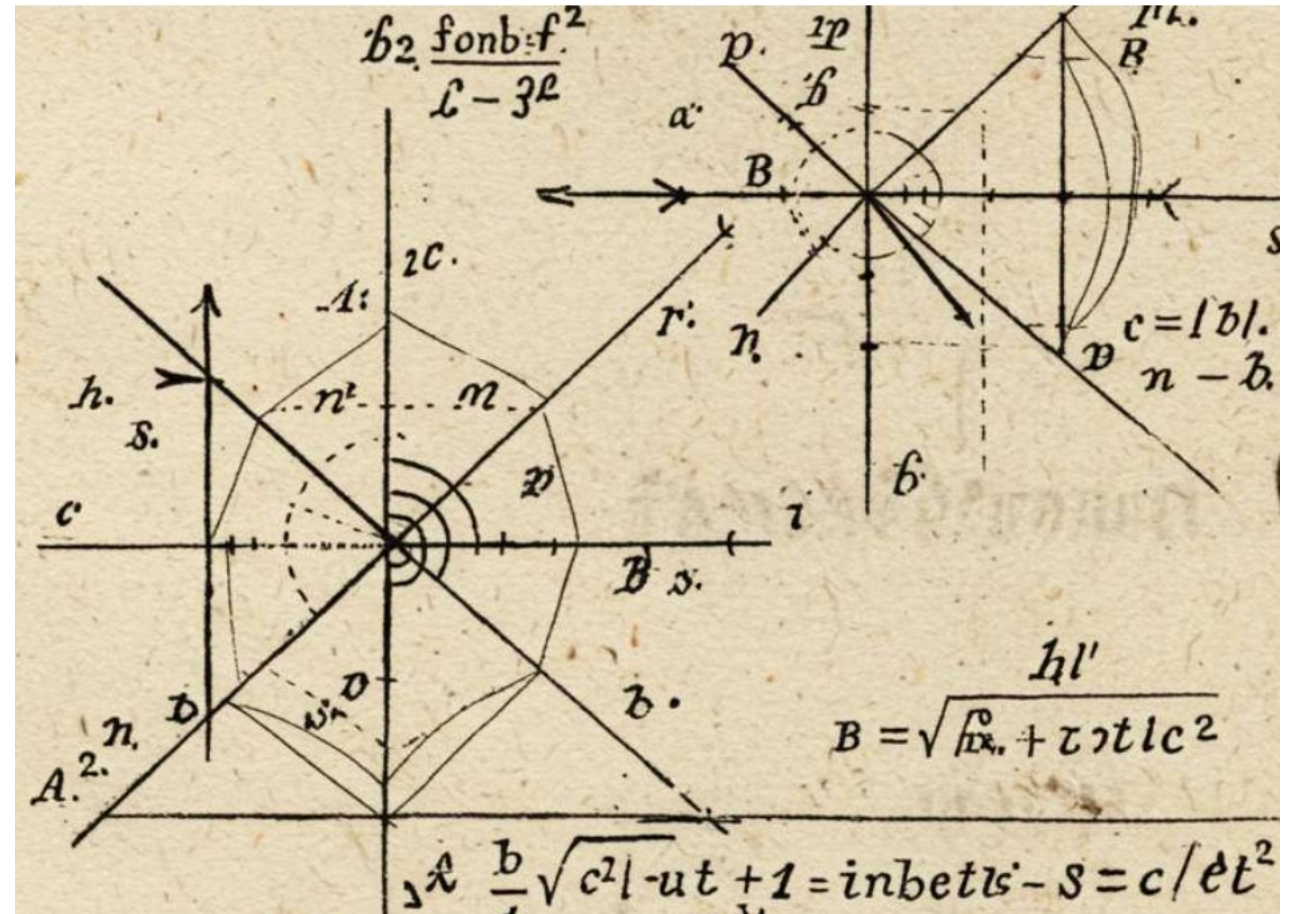
COMPLEX NUMBERS AS GEOMETRY

- In Lecture 2 you saw how we defined an algebra of complex numbers.
 - An algebra is a set of objects and rules for combining those objects.
 - We defined the set of objects to be all combinations of a real number plus an imaginary number.
 - We defined the rules to be addition, multiplication, exponentiation.
 - The objects and rules are represented by discrete symbols. Arithmetic is an algebra where the objects are real numbers, and the rules are things like $+$ and $\times$. The algebra gives us ways to produce new sets of legal symbol combinations.
 - Geometry on the other hand describes the relationships between objects in space. Euclid's geometry writing had no symbols.
 - Rene Descartes was the first to show that the same mathematical objects could be described geometrically (drawings) and algebraically (applying symbol rules).
 - We will find it useful in this course to switch back and forth between the two representations.
-

GEOMETRY IS ALGEBRA



Κέντρω μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πὸ κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπὶ τὰ Α, Β σημεία ἐπεζεύχθω εὐθεῖαι αἱ ΓΑ, ΓΒ.



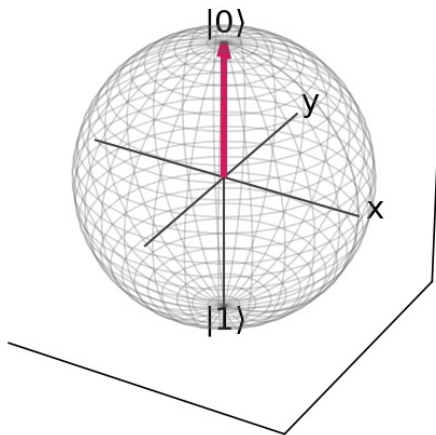
A) **Geometry.** Spatial reasoning only. Symbols are named points, not variables. **Euclid (Elements, ca. 300 BCE)**

B) **Analytic Geometry.** Spatial objects are mapped to algebraic objects. **René Descartes (La Géométrie, 1637)**

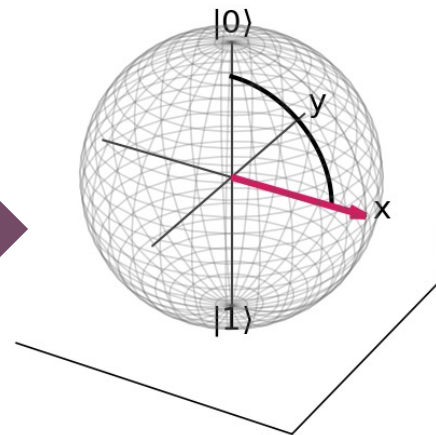
WHY IT MATTERS FOR QUANTUM COMPUTING

- We will apply algebraic operations (gates) to a quantum state and describe what is happening geometrically with Bloch spheres.
- Algebraically: $|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{Z} |-\rangle \xrightarrow{H} |1\rangle$
- Geometrically:

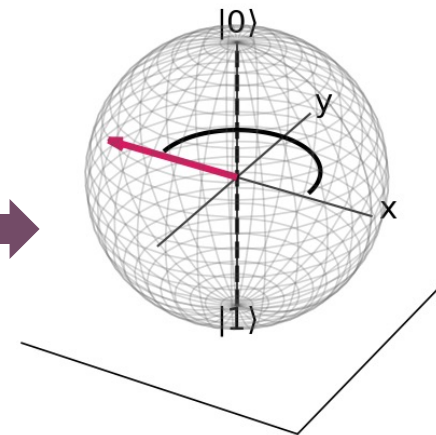
Start



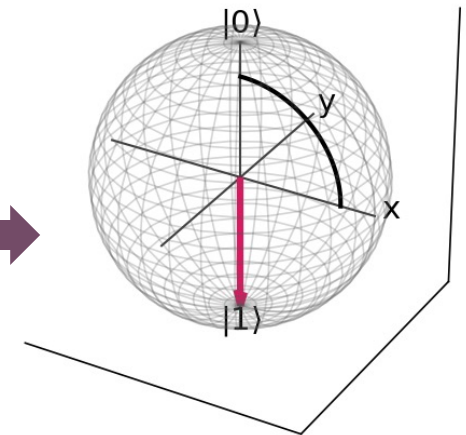
Apply H-gate



Apply Z-gate



Apply H-gate

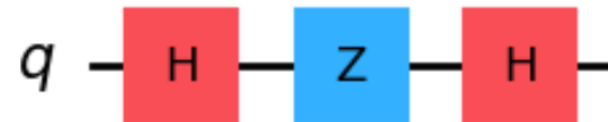


WHY IT MATTERS FOR QUANTUM COMPUTING

- We will apply algebraic operations (gates) to a quantum state and describe what is happening geometrically with Bloch spheres.
- And eventually in code:

```
In [1]: 1 from qiskit import QuantumCircuit
        2
        3 qc = QuantumCircuit(1)
        4
        5 qc.h(0)
        6 qc.z(0)
        7 qc.h(0)
        8
        9 qc.draw('mpl')
```

Out [1]:



(Aside: Programming languages are generally not algebras, but the quantum part of quantum computing is an algebra.)

THE COMPLEX PLANE

a complex number $z = a + bi$ is a point or vector in the plane

this plane is the **simplest possible state space** where algebra already *is* geometry

This mirrors the Bloch sphere idea, but in 2D instead of 3D

Let's work out how how numbers can be manipulated in the complex plane

MULTIPLICATION

- Multiplication in the complex algebra corresponds geometrically to rotation and scaling of vectors in the complex plane.
 - In a quantum computer, the state of the system is described by complex numbers.
 - Programming a quantum computer means applying operations that multiply and combine those complex numbers in precise ways.
-

MULTIPLICATION

A complex number $z = a + bi$ multiplied by complex number $w = c + di$ is,

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Is a **vector** in the complex plane, with:

horizontal component a

vertical component b

So every complex number has:

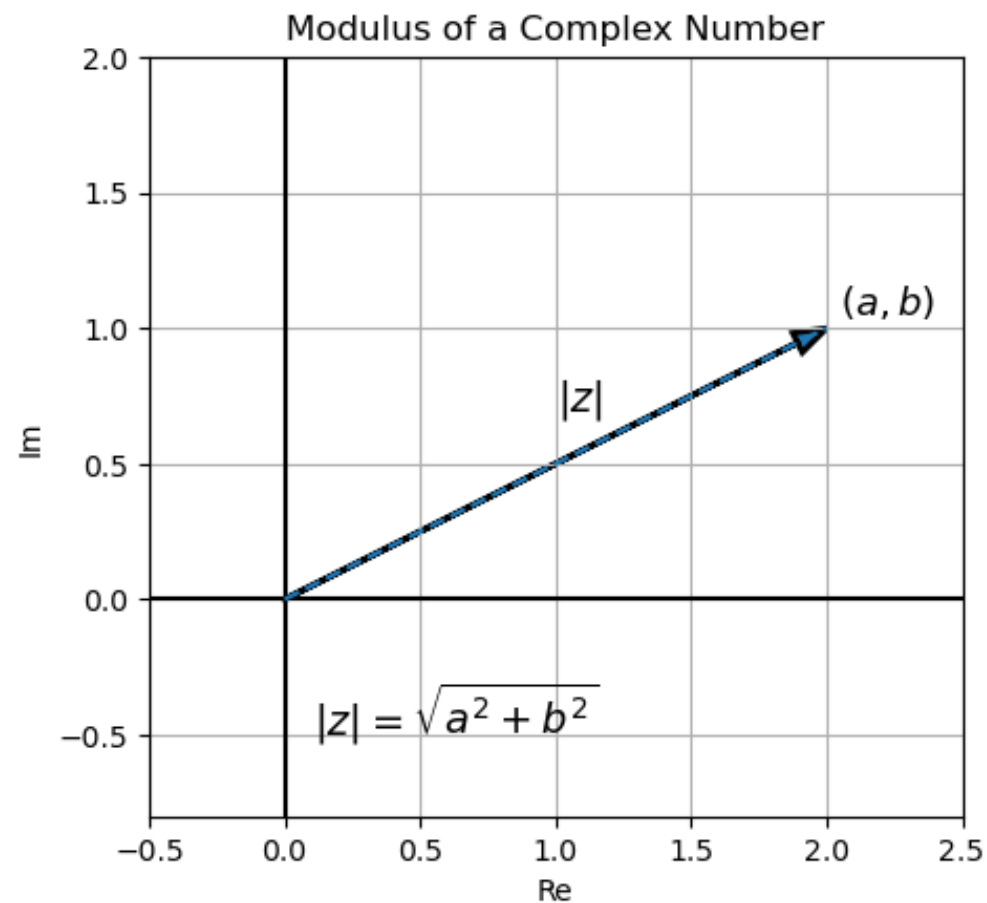
a **length** (its modulus) $|z| = \sqrt{a^2 + b^2}$

a **direction** (its argument) $\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$

MODULUS

A complex number $z = a + bi$

Modulus in vector calculus is just the
length of the vector: $|z| = \sqrt{a^2 + b^2}$



Modulus is a very overloaded term, especially in computer science. It is just Latin for measure.

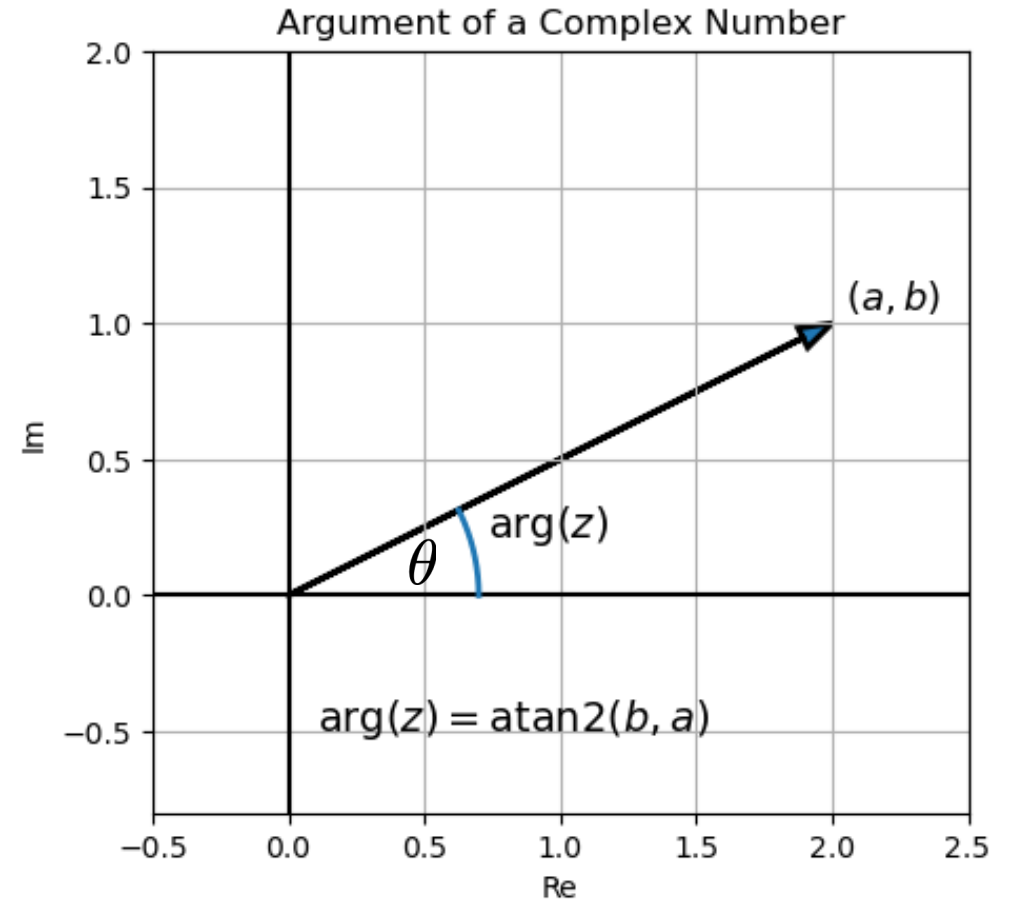
ARGUMENT

A complex number $z = a + bi$

Argument in vector calculus is just angle between the vector and the positive real number line (pos. x-axis).

$$\arg(z) = \tan^{-1} \left(\frac{b}{a} \right) = \theta$$

It's called the *argument* because the standard function for manipulate complex vectors takes θ as an argument.



Algebra

$$z = 1 + 2i \longleftrightarrow (1, 2)$$

$$2z = 2(1 + 2i) = 2 + 4i \longleftrightarrow (2, 4)$$

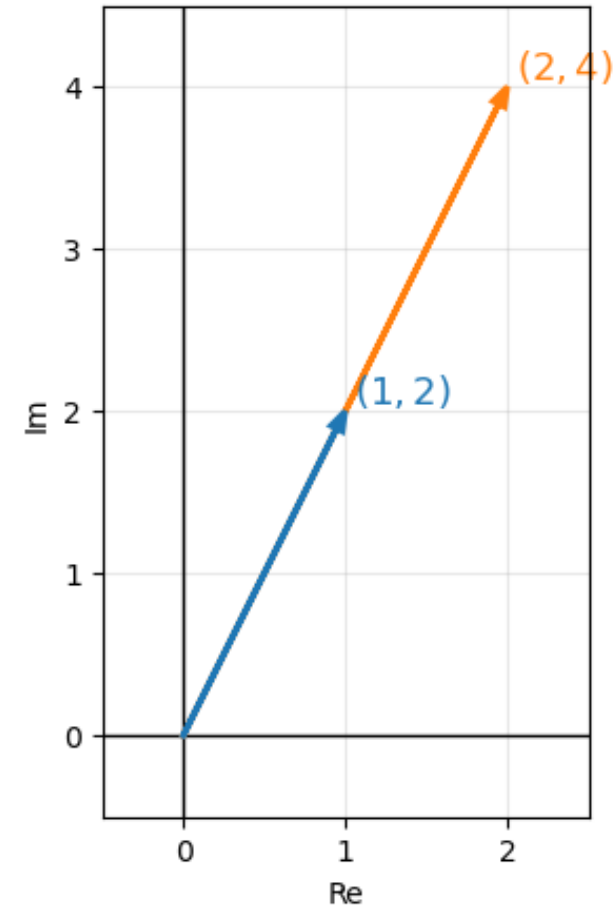
$$|2z| = 2|z|$$

Multiplication of a complex number by a real scalar lengthens the complex vector

REAL SCALAR MULTIPLICATION

Geometry

Scalar multiplication scales a complex vector



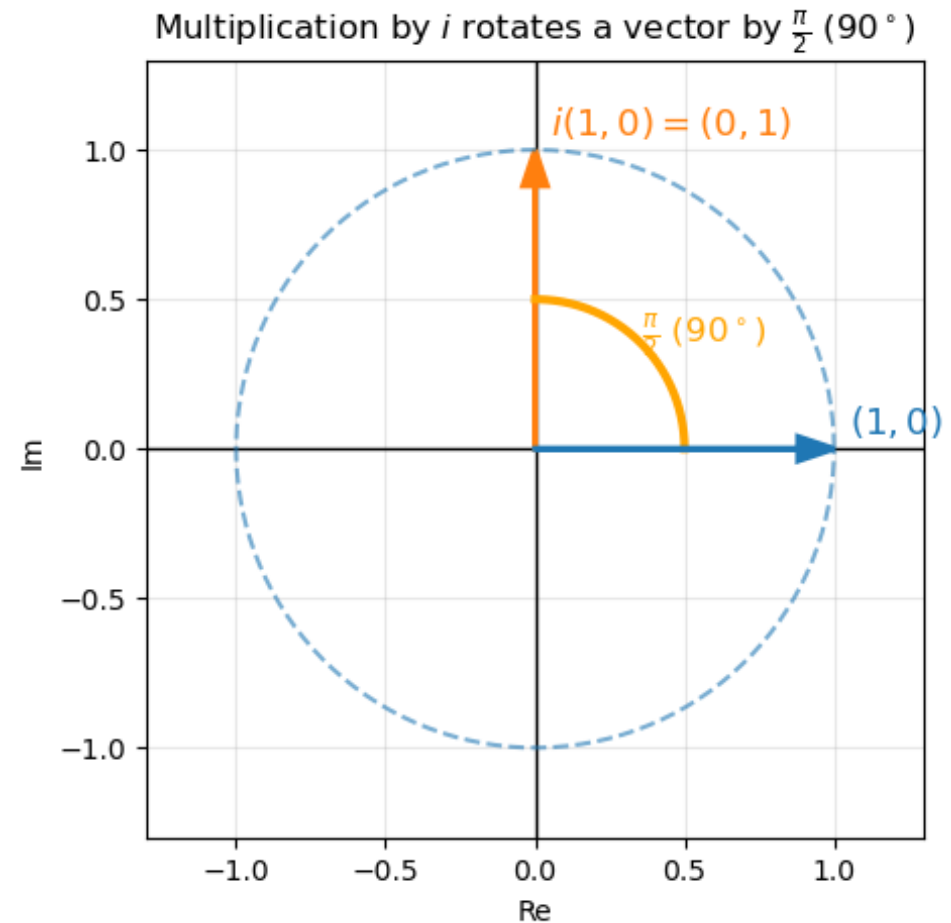
Algebra

Geometry

$$z = 1 + 0i \longleftrightarrow (1, 0)$$
$$iz = i(1 + 0i) = 0 + 1i \longleftrightarrow (0, 1)$$

Multiplication of the real number 1 by the imaginary number i rotates the complex vector by 90 degrees

**MULTIPLICATION
ROTATION ONLY**



Algebra

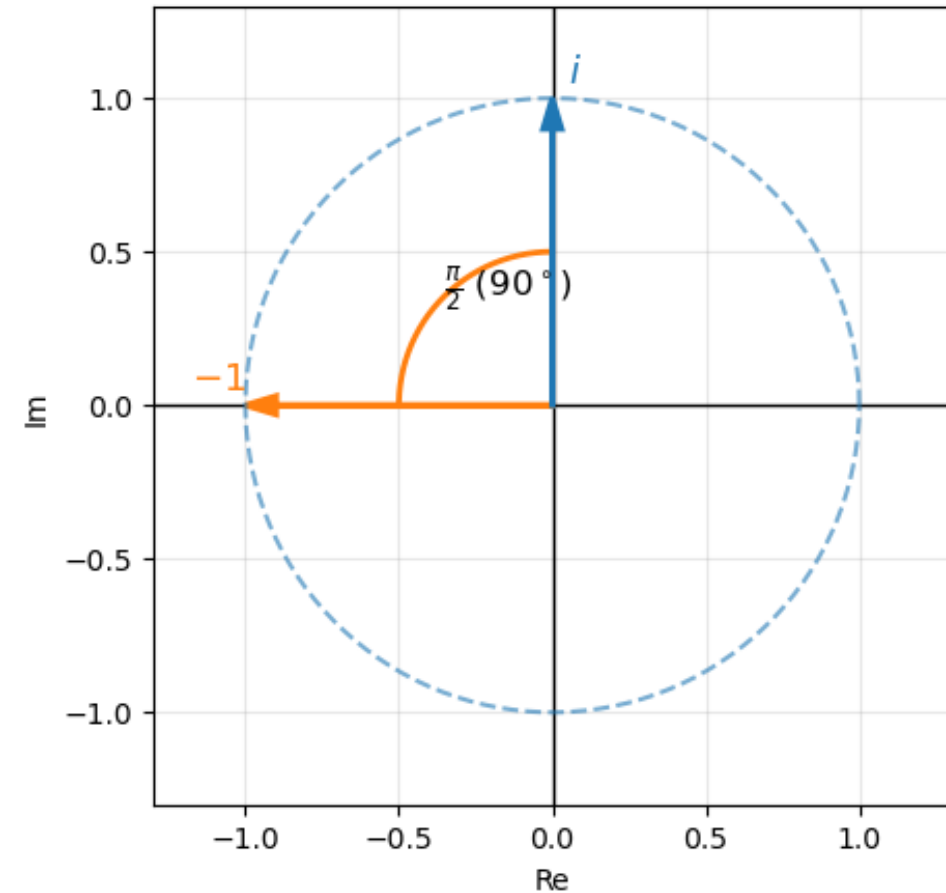
$$0 + i \longleftrightarrow (0, 1)$$
$$(0 + i)(0 + i) = -1 \longleftrightarrow (-1, 0)$$

Multiplication of the imaginary number i by itself rotates the complex vector by 90 degrees

**MULTIPLICATION
ROTATION ONLY**

Geometry

Multiplying i by i rotates i by $\frac{\pi}{2}$ (90°) to -1



Algebra

$$z = (a, b)$$

$$w = (c, d)$$

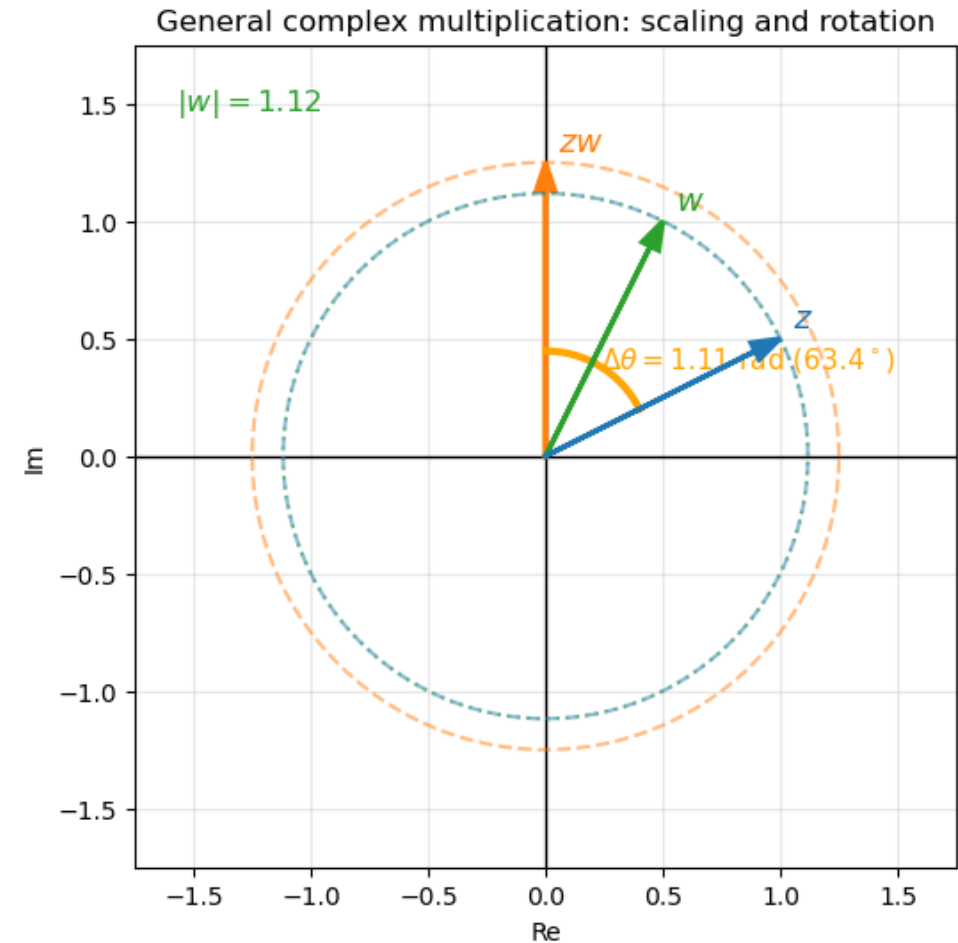
$$\begin{aligned}zw &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

In general multiplying complex vectors together results in a stretched and rotated output vector.

$$|zw| = |w| |z|$$

$$\arg(zw) = \arg(z) + \arg(w)$$

Geometry



Algebra

We have two simple rules:

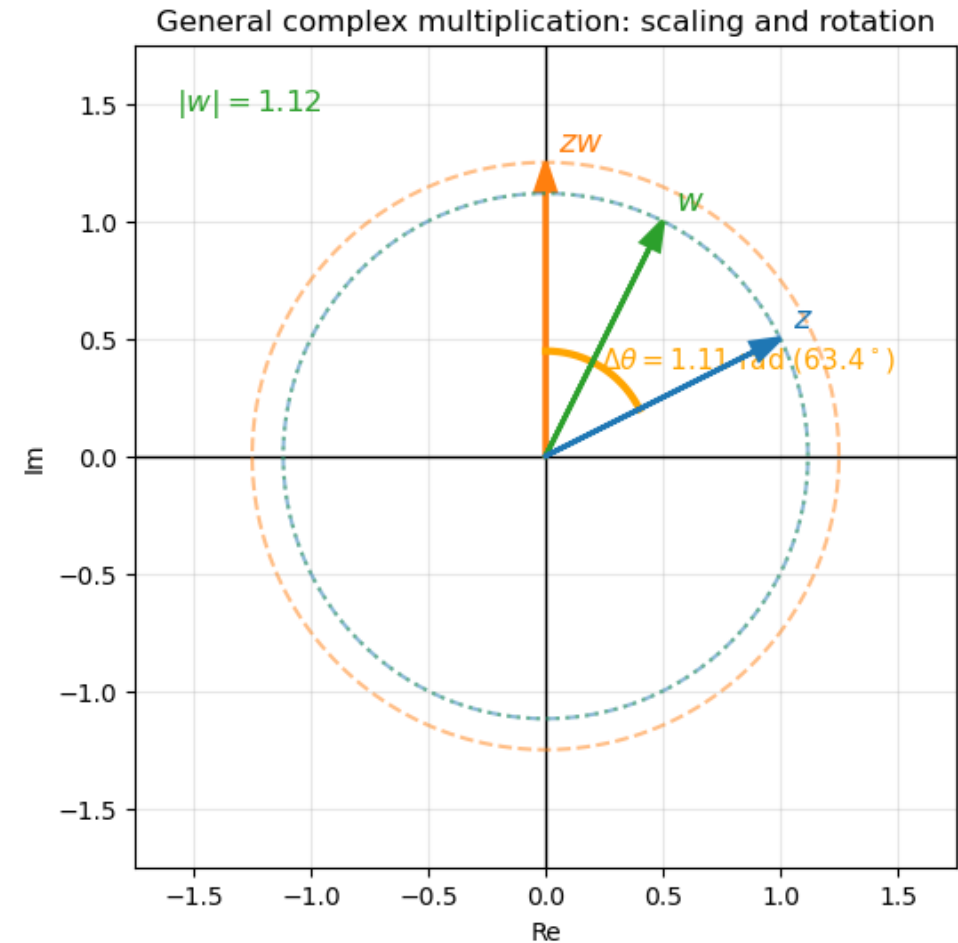
$$|zw| = |w| |z|$$

$$\arg(zw) = \arg(z) + \arg(w)$$

When you multiply two complex numbers, the length of the result is the product of their lengths.

The argument of the product is the sum of their arguments.

Geometry



ALGEBRAIC DERIVATION OF THE MODULUS PRODUCT

RULE

$$\text{Thm: } |z| = |w||z|$$

$$|z| = \sqrt{a^2 + b^2}, \quad |w| = \sqrt{c^2 + d^2}$$

$$|zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2}$$

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2$$

$$|zw|^2 = a^2c^2 - \cancel{2abcd} + b^2d^2 + a^2d^2 + \cancel{2abcd} + b^2c^2$$

$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$= a^2(c^2 + d^2) + b^2(c^2 + d^2)$$

ALGEBRAIC DERIVATION OF THE MODULUS PRODUCT RULE

$$|z| = \sqrt{a^2 + b^2}, \quad |w| = \sqrt{c^2 + d^2}$$

$$|zw|^2 = (a^2 + b^2)(c^2 + d^2)$$

$$a^2 + b^2 = |z|^2, \quad c^2 + d^2 = |w|^2$$

$$|zw|^2 = |z|^2 |w|^2$$

QED

ALGEBRAIC DERIVATION OF THE ARGUMENT PRODUCT RULE

$$\text{Thm: } \arg(wz) = \arg(w) + \arg(z)$$

$$z = a + bi, \quad w = c + di$$

$$wz = (ac - bd) + (ad + bc)i$$

$$\arg(z) = \arctan\left(\frac{b}{a}\right), \quad \arg(w) = \arctan\left(\frac{d}{c}\right)$$

$$wz = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$\arg(x + yi) = \arctan\left(\frac{y}{x}\right)$$

$$\arg(wz) = \arctan\left(\frac{ad + bc}{ac - bd}\right)$$

ALGEBRAIC DERIVATION OF THE ARGUMENT PRODUCT RULE

$$\text{Thm: } \arg(wz) = \arg(w) + \arg(z)$$

$$\tan(\arg(wz)) = \frac{ad + bc}{ac - bd}$$

$$\text{Trig identity } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\arg z + \arg w) = \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}}.$$

ALGEBRAIC DERIVATION OF THE ARGUMENT PRODUCT RULE

$$\text{Thm: } \arg(wz) = \arg(w) + \arg(z)$$

$$\tan(\arg z + \arg w) = \frac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{bd}{ac}}.$$

$$\tan(\arg z + \arg w) = \frac{\frac{b}{a} + \frac{d}{c}}{\frac{ac}{ac} - \frac{bd}{ac}} = \frac{\frac{bc+ad}{ac}}{\frac{ac-bd}{ac}}$$

$$\frac{\frac{bc+ad}{ac}}{\frac{ac-bd}{ac}} = \frac{bc+ad}{ac} \cdot \frac{ac}{ac-bd} = \frac{ad+bc}{ac-bd}$$

ALGEBRAIC DERIVATION OF THE ARGUMENT PRODUCT RULE

$$\text{Thm: } \arg(wz) = \arg(w) + \arg(z)$$

$$\tan(\arg w + \arg z) = \frac{ad + bc}{ac - bd}$$

AND

$$\tan(\arg(wz)) = \frac{ad + bc}{ac - bd}$$

QED

COMPLEX MULTIPLICATION

- Now we have a way to relate any two vectors in the complex plane:

$\left| \frac{w}{z} \right|$ tells you the difference in length

$\arg \left(\frac{w}{z} \right)$ tells you the difference in rotation

We are thinking of an algebraic and geometric objects.

$$z = a + bi \in \mathbb{C}, \quad \vec{z} = (a, b) \in \mathbb{R}^2$$

POWERS

- Exponentiation is repeated multiplication.

- We already know a couple of things from properties of multiplication:

Each multiplication by z scales by $|z|$ $|z^n| = |z|^n$

Each rotation by z adds $\arg(z)$ $\arg(z^n) = n \arg(z) \pmod{2\pi}$

Yes, these are both modulus:
Mathematicians are bad at
inventing new names



We don't have a general formula for exponentiation yet, we will move to polar coordinate for that later

POWERS

Sanity checks:

$\forall z \in \mathbb{C}, \forall w \in \mathbb{C} : |w| = 1, wz$ rotates $\vec{z}$ about the origin, resulting in $\overrightarrow{wz}$.

$\forall z \in \mathbb{C}, \forall w \in \mathbb{C} : |w| < 1, wz$ rotates $\vec{z}$ about the origin and shortens, resulting in $\overrightarrow{wz}$.

$\forall z \in \mathbb{C}, \forall w \in \mathbb{C} : |w| > 1, wz$ rotates $\vec{z}$ about the origin and lengthens, resulting in $\overrightarrow{wz}$.

Quantum evolution preserves length, so only rotations are allowed.

INVERSE OF POWERS $\mathbb{R}$

Real numbers have two senses of inverse:

Fixed exponent, solve for the base $x^n = y \Rightarrow x = y^{1/n}$

This is taking the n th root.

Fixed base, solve for the exponent: $a^x = y \Rightarrow x = \log_a(y)$

These are functions: $f : X \rightarrow Y$ $f(x) = x^2 : \mathbb{R} \rightarrow \mathbb{R}$

INVERSE OF POWERS $\mathbb{C}$

Roots and logarithms of complex numbers have more than one possible value.

The multi-valued nature of roots is fundamentally because complex numbers live in a plane, not a line like the real numbers do.

On the real number line there is a strict ordering and no angles between values.

Values on the complex plane have an argument (angle) and modulus (length).

Since we have angles between numbers and angles wrap (mod 2π) then we can have several output values

Logs and roots of complex numbers are therefore not single-valued functions; they are set-valued mappings, where the output is a subset (an element of the powerset of inverses) of complex numbers consisting of all possible inverses.

$$F : X \rightarrow \mathcal{P}(Y)$$

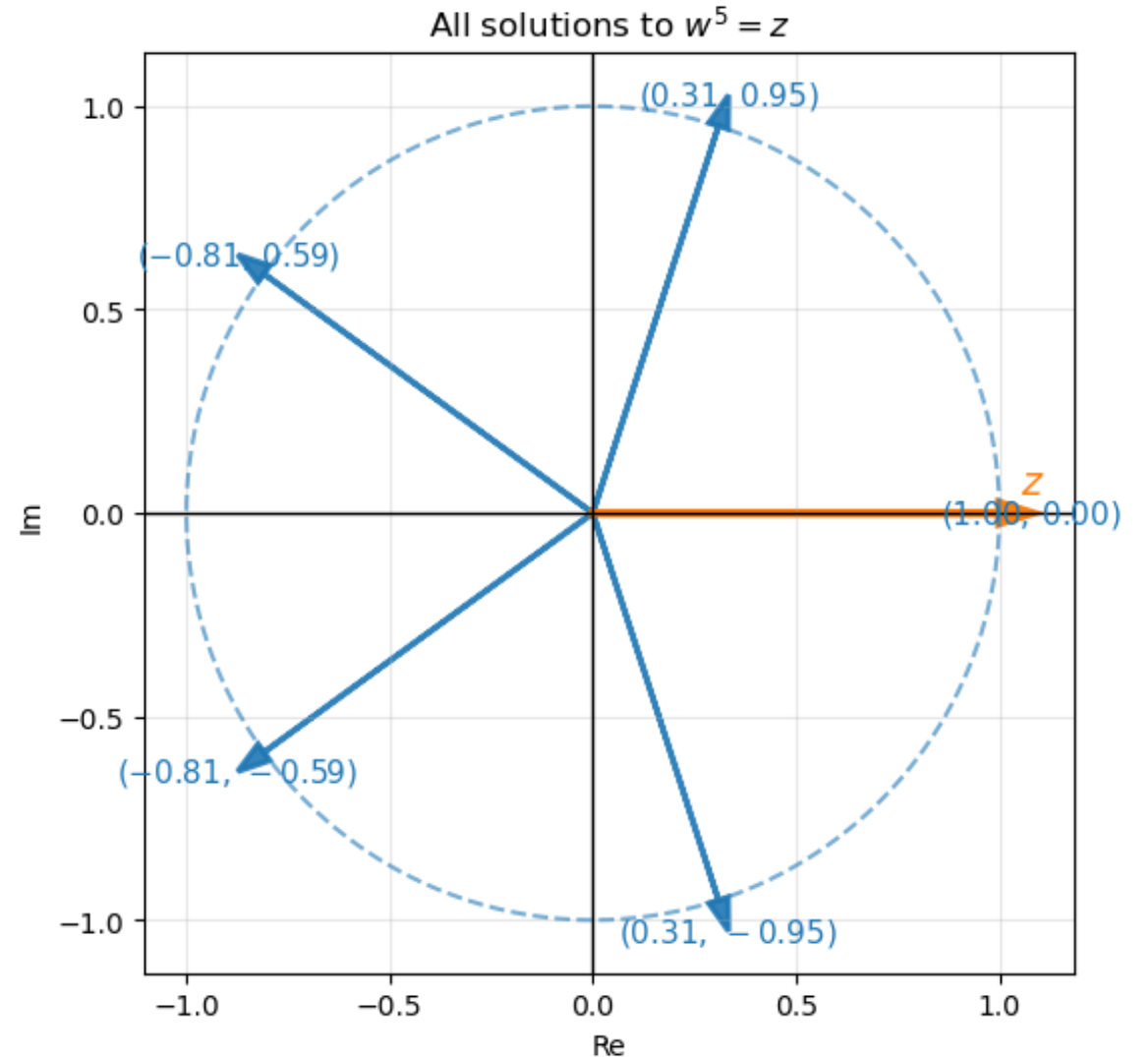
INVERSE OF POWERS $\mathbb{C}$

There are 5 solution vectors to the problem of finding the fifth root of z (4 blue and one under the orange vector.)

We have been using cartesian algebraic representations for the geometry of the complex plane.

Since length and rotation (modulus and argument) are how we are thinking of complex numbers there is another way of mapping geometry to algebra: **polar coordinates**.

$$z = r(\cos \theta + i \sin \theta), \quad r = |z|, \quad \theta = \arg(z)$$

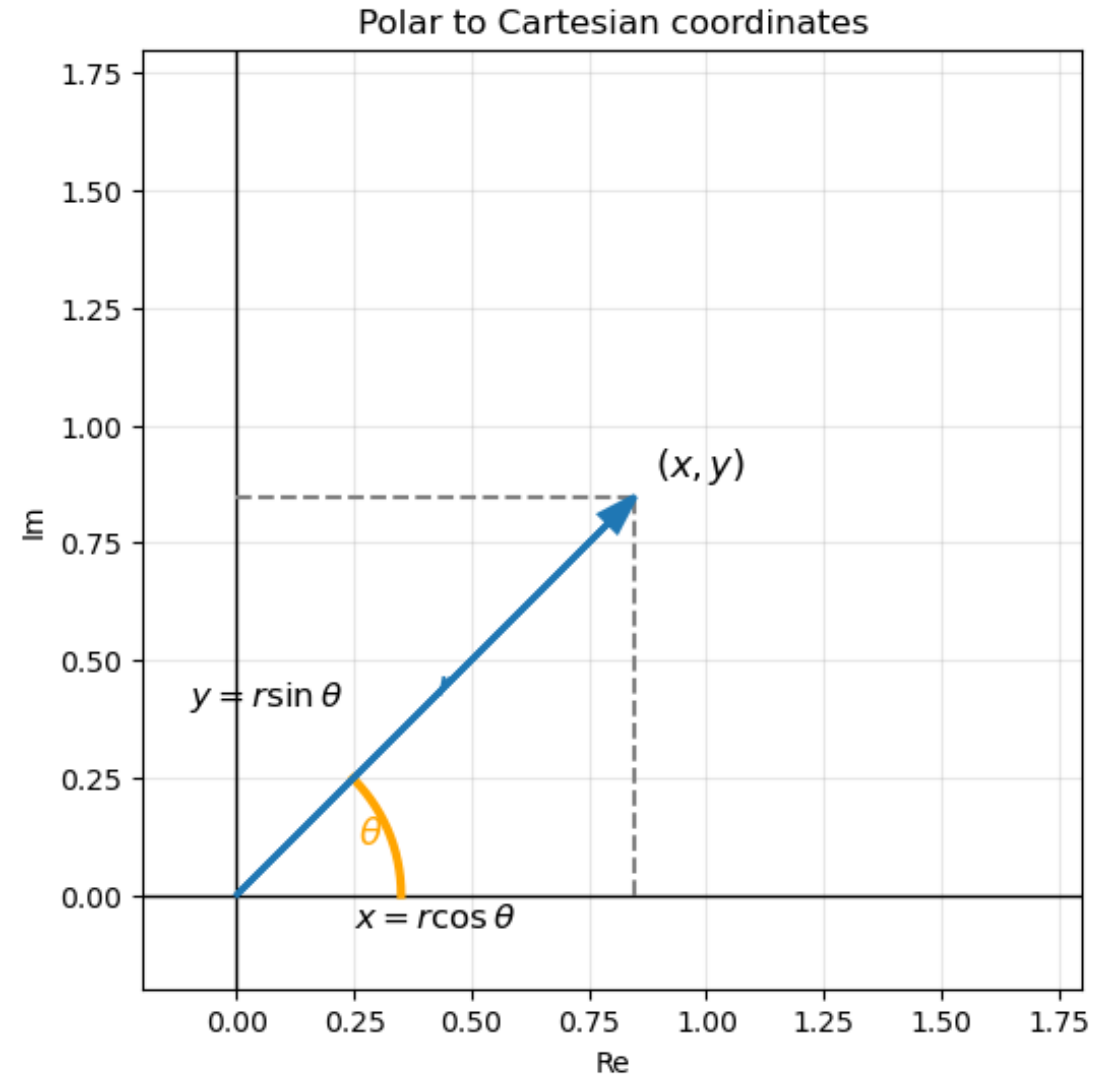


INVERSE OF POWERS $\mathbb{C}$

$$z = r(\cos \theta + i \sin \theta), \quad r = |z|, \quad \theta = \arg(z)$$

The mapping between polar and cartesian coordinates is:

$$(x, y) = (r \cos \theta, r \sin \theta)$$



MULTIPLICATION IN $\mathbb{C}$

Cartesian form:

$$w = a + bi, \quad z = c + di$$
$$wz = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Polar form:

$$w = r_w(\cos \theta_w + i \sin \theta_w), \quad z = r_z(\cos \theta_z + i \sin \theta_z)$$
$$wz = r_w r_z (\cos(\theta_w + \theta_z) + i \sin(\theta_w + \theta_z))$$

In Cartesian form, multiplication produces terms like ac and ad , whose geometric meaning is not immediately clear.

In polar form, multiplication corresponds directly to multiplying lengths and adding angles, which gives a clear geometric interpretation.

POWERS IN $\mathbb{C}$

$$z = r(\cos \theta + i \sin \theta), \quad r = |z|, \quad \theta = \arg(z)$$

Since complex multiplication in polar form is

$$wz = r_w r_z (\cos(\theta_w + \theta_z) + i \sin(\theta_w + \theta_z))$$

Exponentiation is just:

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

Exponentiation scales the length by r^n and multiplies the angle by n .

POWERS IN $\mathbb{C}$

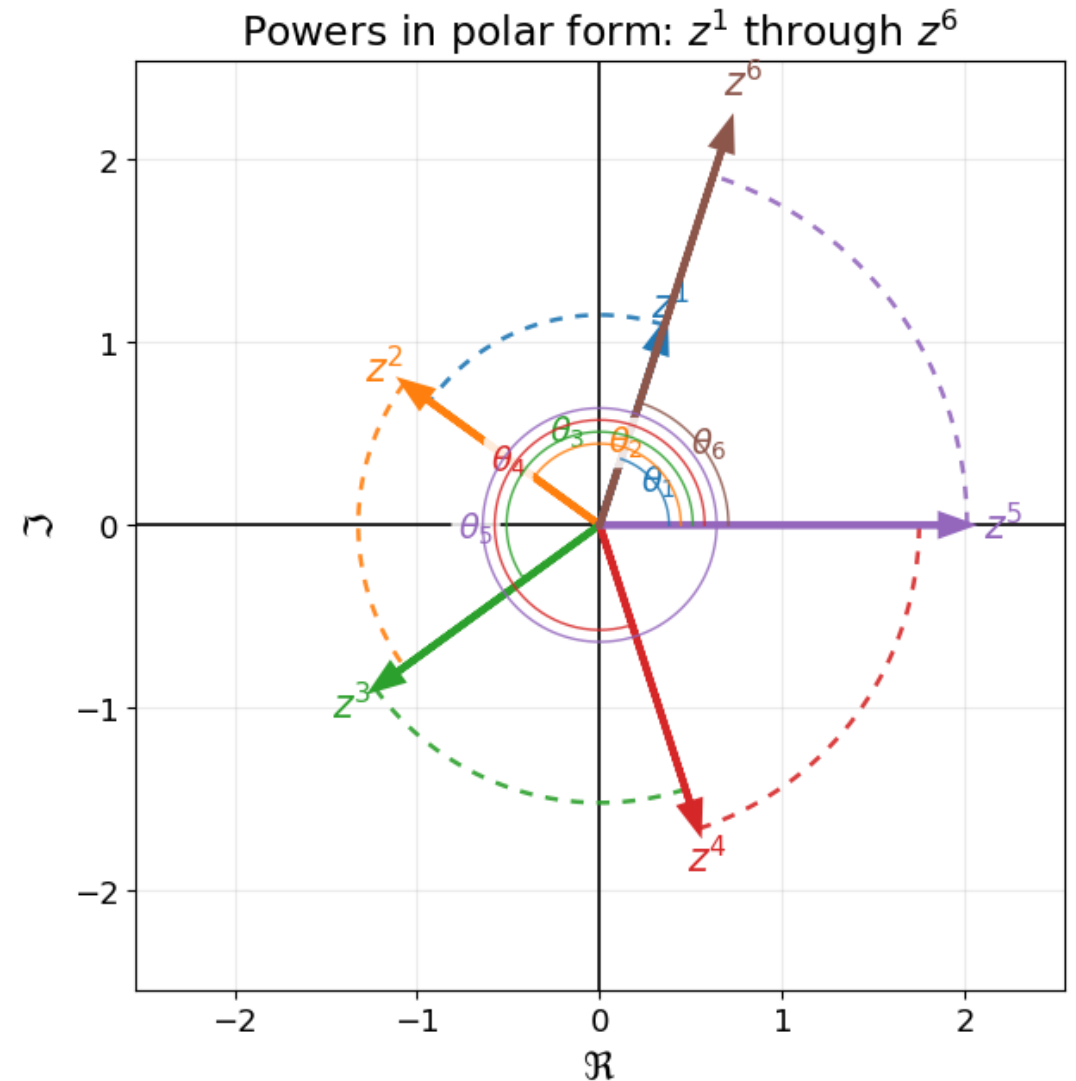
Exponentiation is just:

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

Exponentiation scales the length by r^n and multiplies the angle by n .

The modulus and argument of the vector result of each successive power increases.

This is De Moivre's formula



POWERS IN $\mathbb{C}$

De Moivre's formula

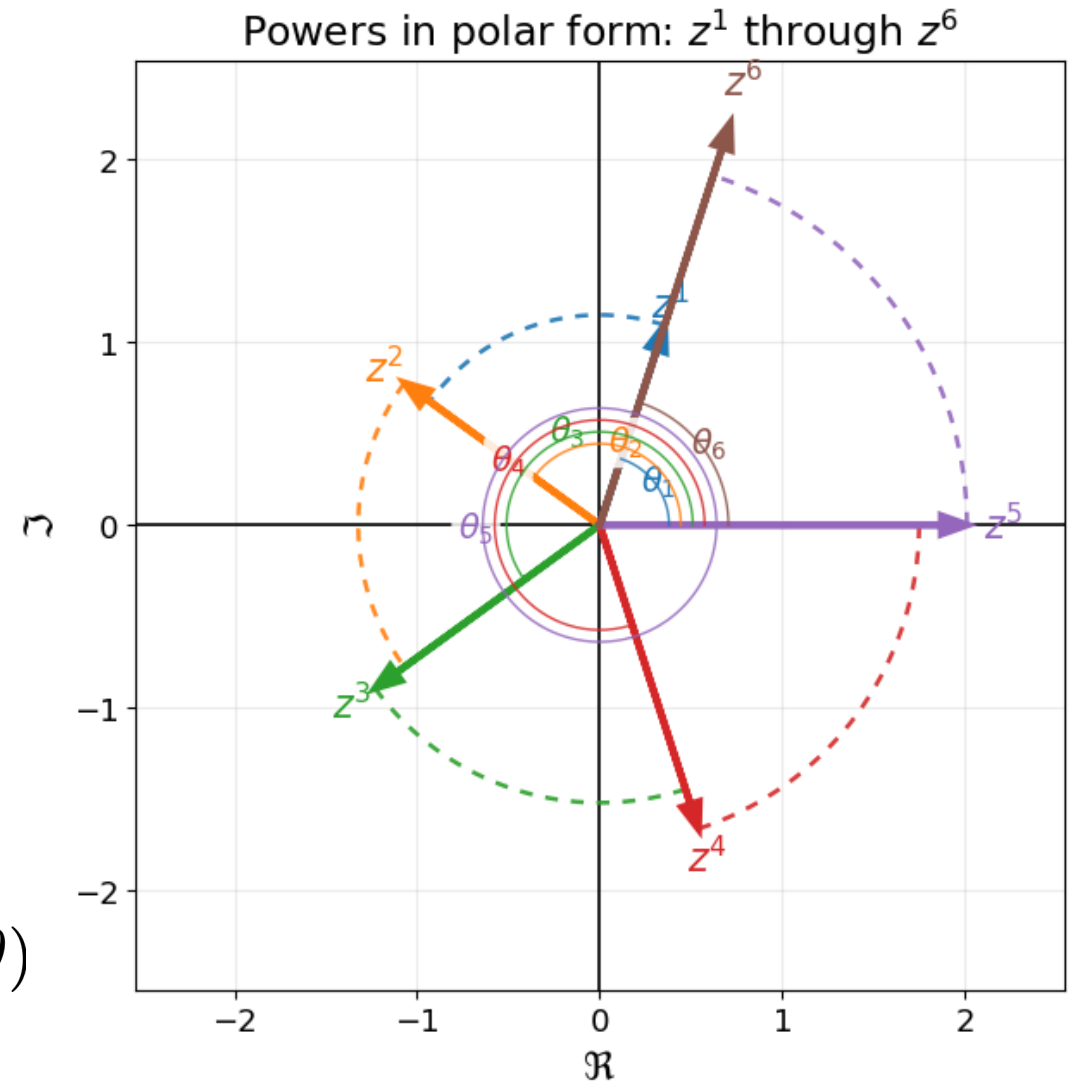
Exponentiation is just:

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta))$$

In Quantum Mechanics we restrict the vectors to have modulus 1. I.e they define a unit circle.

In which case the formula can drop the modulus

$$z^n = (\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$$



POWERS IN $\mathbb{C}$

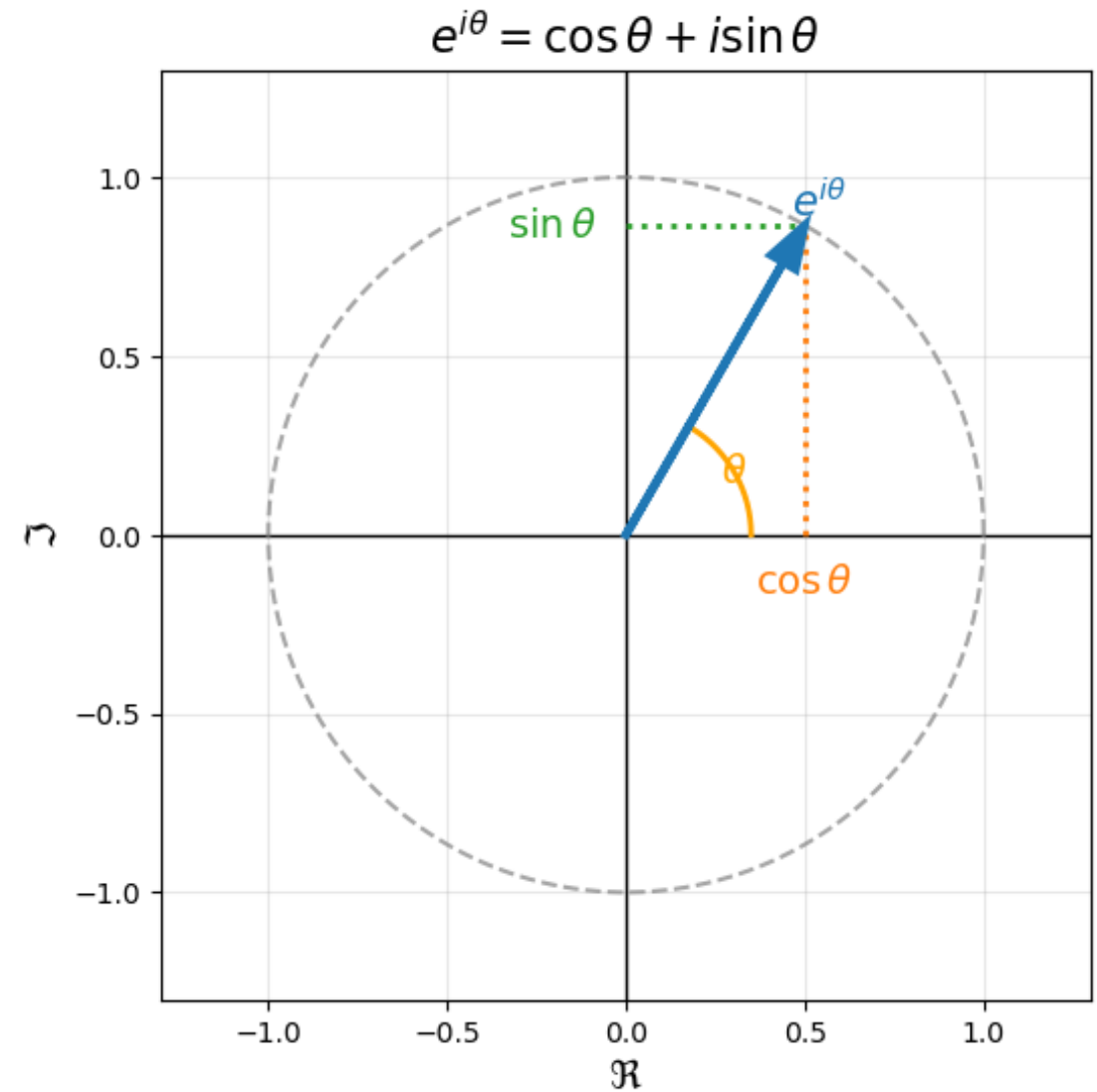
Euler's formula

De Moivre tells us what happens when we raise a rotation to an integer power.

Euler's formula tells us how to describe rotation itself.

The following is how Euler proved it. But proofs are just an argument that convinces your reader.

The following proof is no longer considered rigorous.



POWERS IN $\mathbb{C}$

Euler's Proof* (1740s)

Using the series representations of e^x , $\cos \theta$, and $\sin \theta$,
we rearrange terms to obtain $\cos \theta + i \sin \theta = e^{i\theta}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \cos \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}, \sin \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

Series definitions

$$\cos \theta + i \sin \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

Substitute series

$$\cos \theta + i \sin \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(-1)^n \theta^{2n+1}}{(2n+1)!}$$

Distribute i into second sum

$$\cos \theta + i \sin \theta = \sum_{n=0}^{\infty} \frac{(i\theta)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\theta)^{2n+1}}{(2n+1)!}$$

Use $i^{2n}=(-1)^n$

$$\cos \theta + i \sin \theta = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!}$$

Combine even and odd terms

$$\cos \theta + i \sin \theta = e^{i\theta}$$

Well, that's interesting!

*Euler predates Karl Weierstrass, whose development of real analysis established modern standards of rigor for proofs involving infinite processes.

PHASE (PREVIEW)

- In quantum computing, the angle (argument) of a complex number is called its *phase*.
- Some of the properties of quantum computing that are most powerful depend on the relative phase of a qubit:

$$\psi = a |0\rangle + b |1\rangle,$$

- The relative phase of the qubit's components a and b is given by:

$$\arg(b) - \arg(a)$$
