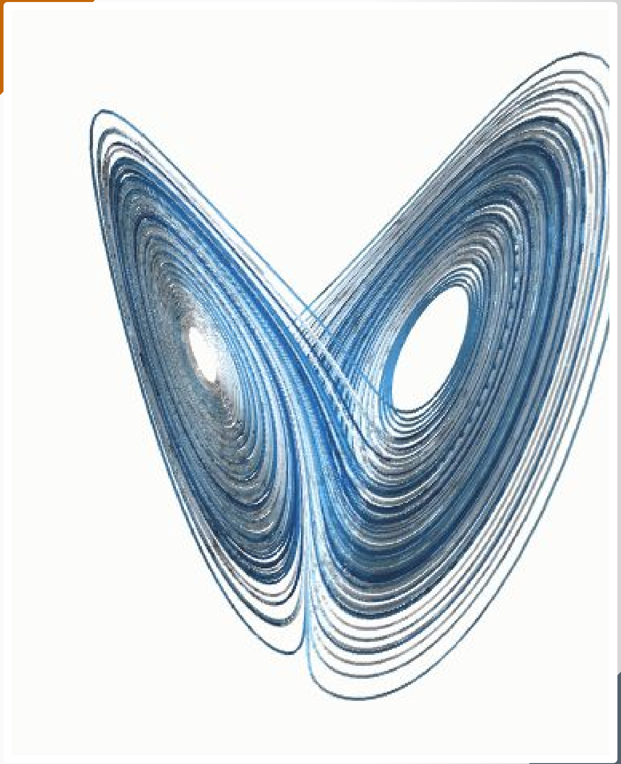

Computational Chaos by Edward N. Lorenz

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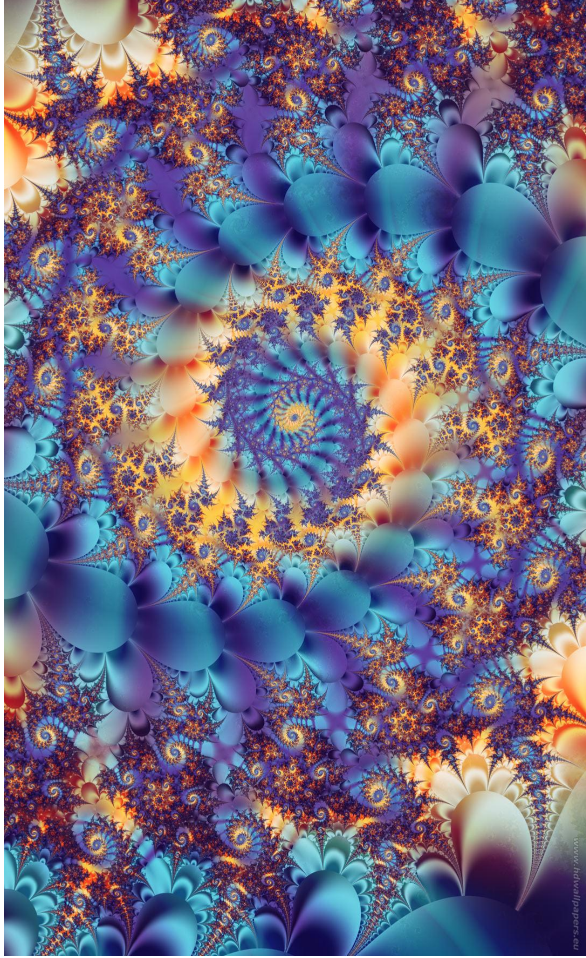
Outline

- Introduction
- A system with fixed-point attractors
- Instability of fixed points
- Bifurcations
- Onset of chaos
- Strange Attractors
- Conclusion



Introduction

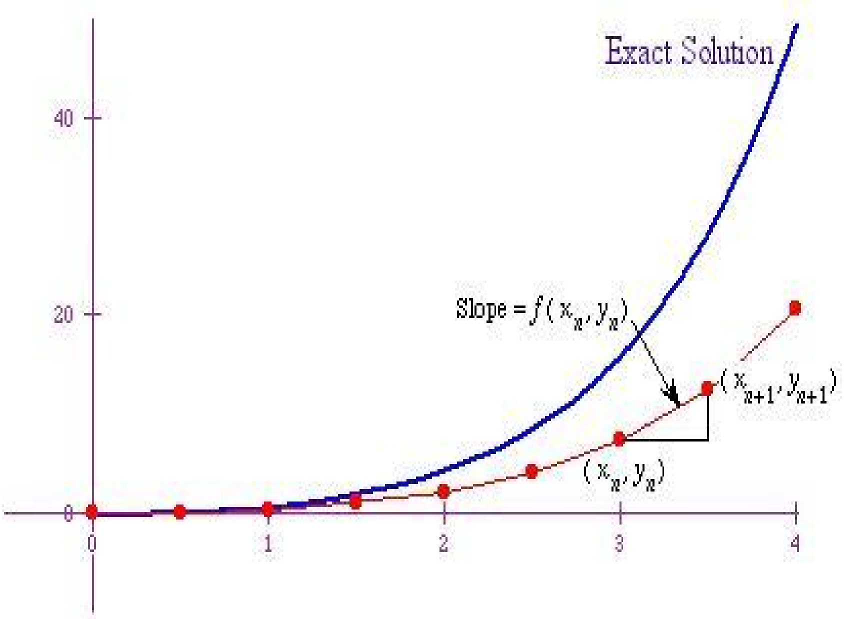
- Solutions to nonlinear differential equations often sought by numerical means.

$$\mathrm{d}\boldsymbol{X}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{X})$$

- Approximating using Euler scheme:

$$\boldsymbol{X}_{n+1} = \boldsymbol{X}_n + \tau \boldsymbol{F}(\boldsymbol{X}_n)$$

- Other techniques:
  - Runge-Kutta
  - 4th-Order Taylor Series



Introduction

- Chaotic behavior sometimes occurs when difference equations used as approximation to ordinary differential equation are solved numerically with a large time increment.

- Using one example we show when fixed points go unstable and when chaos first sets in

Introduction

- **Computational chaos**: A chaotic behavior that owes its existence to the use of a large time increment, $\boldsymbol{\tau}$.
- **Goal:**
  - Lower limit of values of τ for which fixed points are unstable?
  - Lower limit of values of τ for which chaos is present?

Introduction

- Computational chaos is widespread: Even for one of the simplest flows

$$dx/dt = x - x^2$$

- Almost all solutions approach either -∞ or the stable fixed point x = 1
- If we apply Euler function:

$$x_{n+1} = (1+\tau)x_n - \tau x_n^2$$

- $0 < x_0 < (1+\tau)/\tau$ ⇒ fixed point x = 1 if τ < 2

  ⇒ chaotic behavior if 2 < τ < 3

A system with fixed-point attractors

- Used a model of fluid convection and turned into a limiting form of it.

$$\mathrm{d}X/\mathrm{d}t = -\sigma X + \sigma Y,$$
$$\mathrm{d}Y/\mathrm{d}t = -XZ + \rho X - Y,$$
$$\mathrm{d}Z/\mathrm{d}t = XY - \beta Z,$$

$$\Longrightarrow$$

$$\mathrm{d}x/\mathrm{d}t = ax - xy,$$
$$\mathrm{d}y/\mathrm{d}t = -y + x^2$$

- As: $\sigma \longrightarrow \infty$, $\beta = 1$, a = ϱ -1
- Replace X by Y
- Replace Y and Z by x and y

A system with fixed-point attractors

- Approximating the limiting form by Euler scheme:

$$x_{n+1} = (1 + a\tau)x_n - \tau x_n y_n$$

$$y_{n+1} = (1 - \tau)y_n + \tau x_n^2.$$

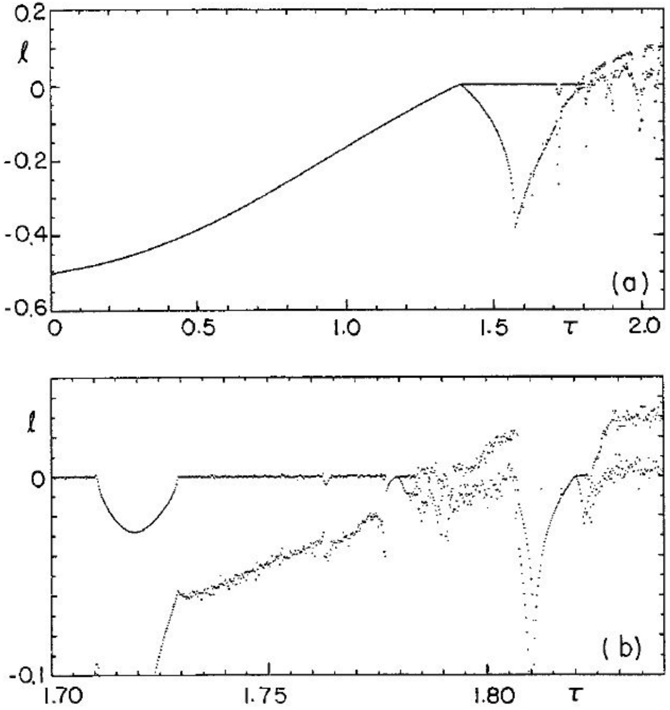
- After solving the differential equation we get Lyapunov exponents: l_1 and l_2 which are both equal to:

$$[\log(1 - \tau + 2a\tau^2)]/(2\tau),$$

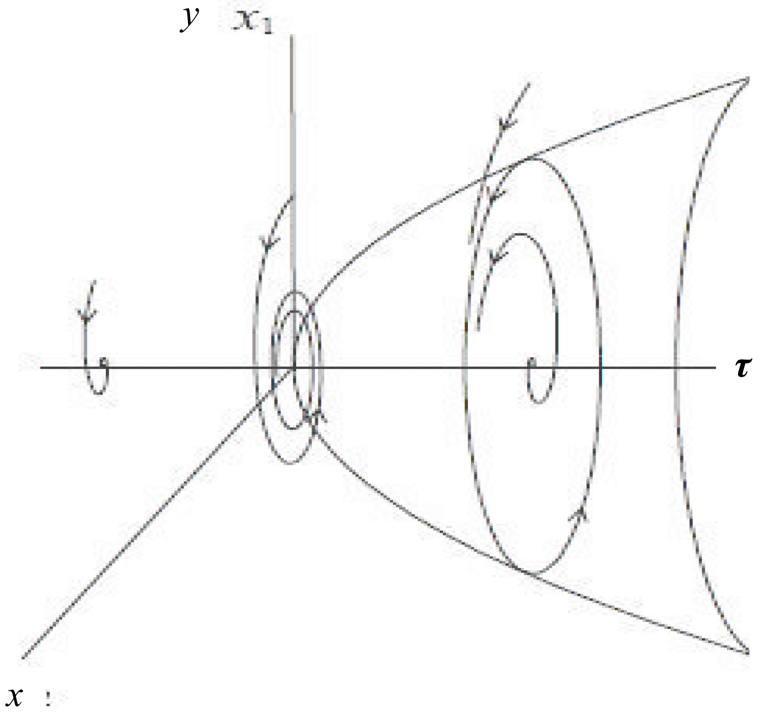
  - If $\boldsymbol{\tau} \to 0$ ⇒ -½ (stable fixed point)
  - $\boldsymbol{\tau}$ = 1/(2a) ⇒ 0

Lyapunov Exponent VS Time Increment

- $0 \leq \boldsymbol{\tau} < \boldsymbol{\tau}$a, $l < 0 \Rightarrow$ Stable fixed point
  - Single stable fixed point
- $\boldsymbol{\tau}$a $\leq \boldsymbol{\tau} < \boldsymbol{\tau}$b, $l > 0 \Rightarrow$ Unstable fixed point
  - Hopf bifurcation
- $\boldsymbol{\tau}$b $\leq \boldsymbol{\tau} < \boldsymbol{\tau}$c, $l > 0 \Rightarrow$ Unstable fixed point
  - Chaos is present
- $\boldsymbol{\tau}$c $\leq \boldsymbol{\tau}$
  - Computational instability



Hopf Bifurcation



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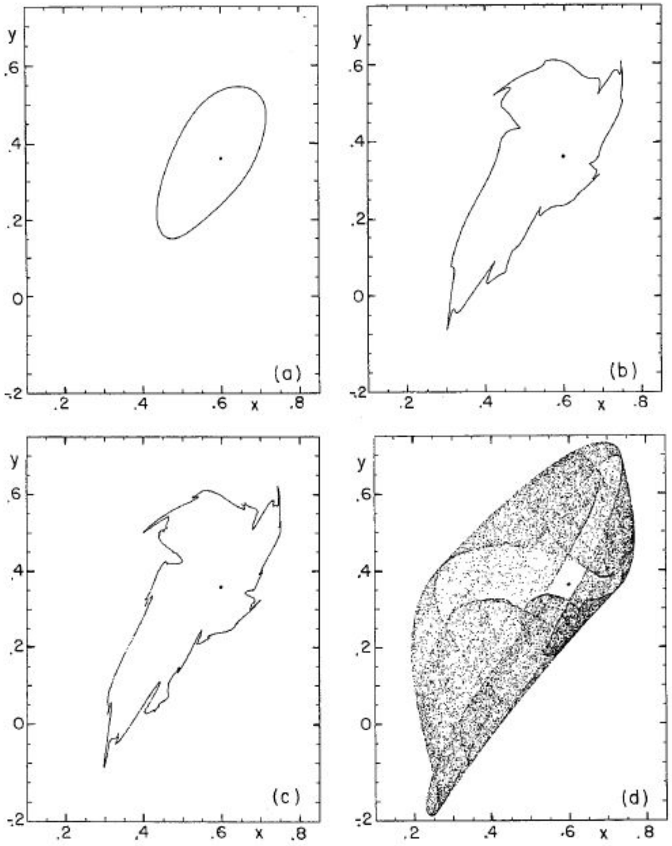
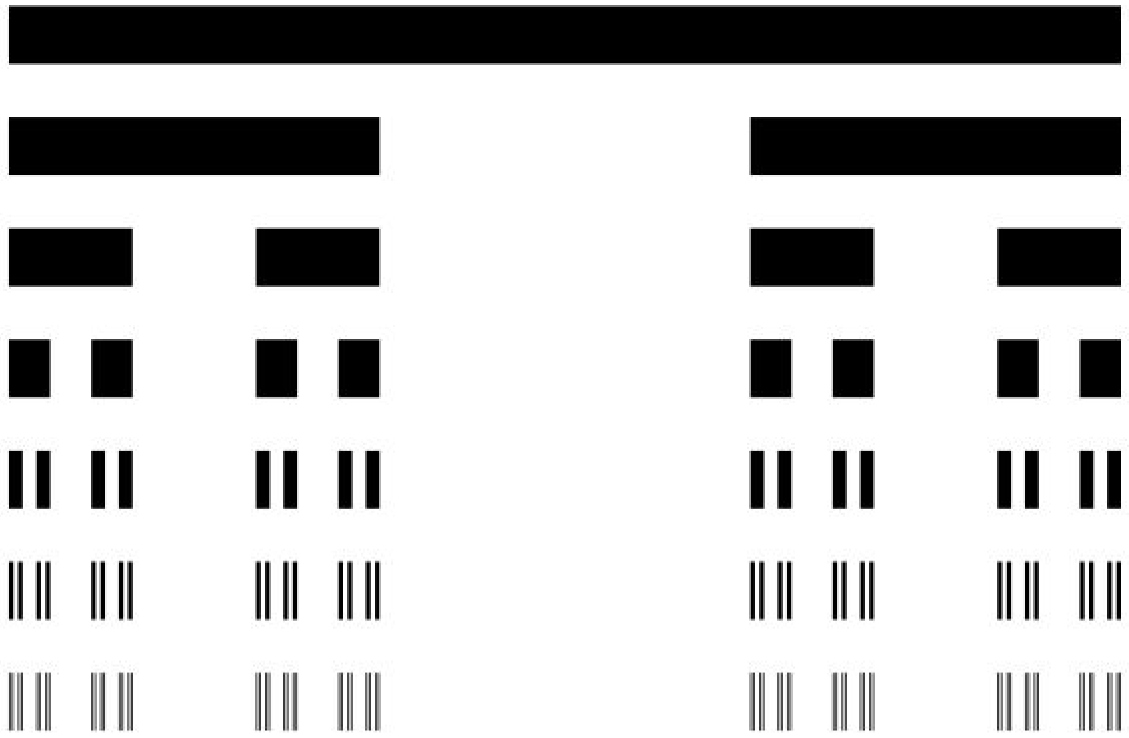
Attractors About A Fixed Point

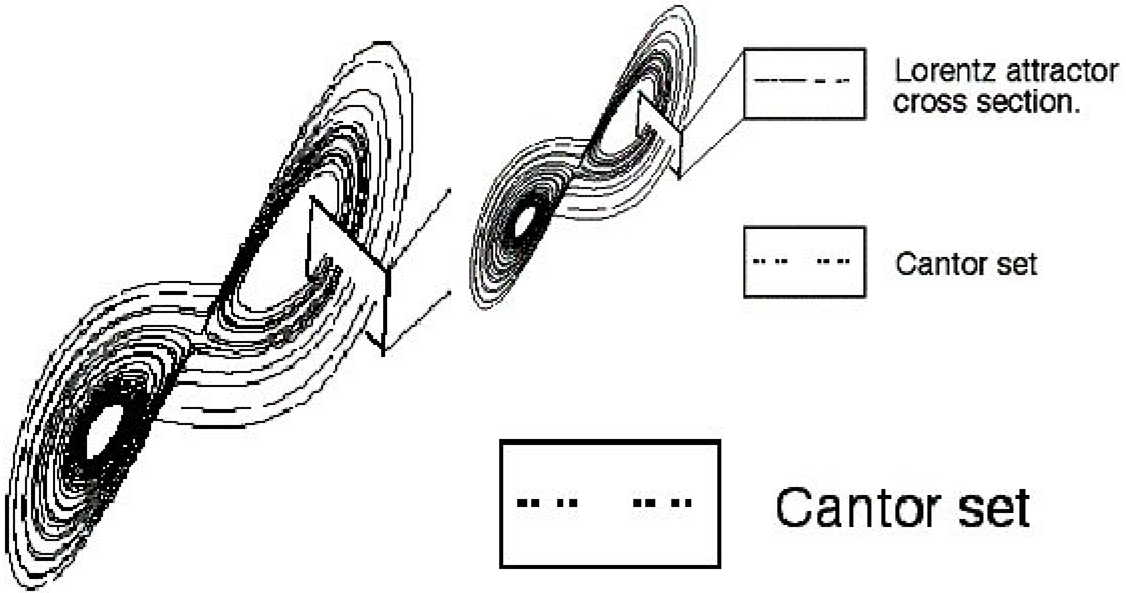
- Limit cycle at Q. Not chaotic
- Deformed limit cycle. Not chaotic
- Further deformed. Chaotic
  - Hopf Bifurcations at kinks



Cantor Set

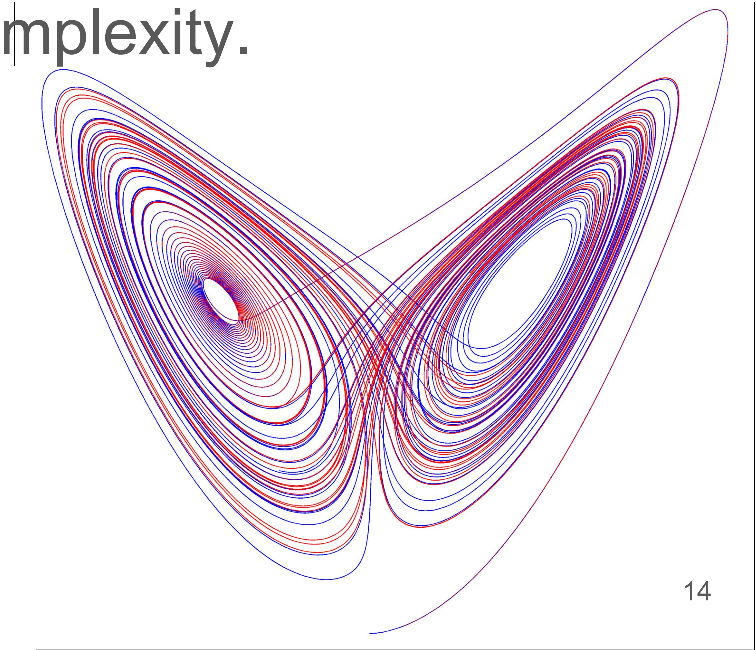


Lorenz Attractor Cross Section



Strange Attractor

- Closed set
- Can be visualized in phase space
- Fractal substructure with infinite complexity.
- Basin of attraction



Conclusion

- Interaction between parameters (a, b, τ) can lead to chaos
- Chaos starts with different types of bifurcation
- Not all approximation techniques are created equal
    - Runge-Kutta has a larger range of τ before chaos but can still go chaotic
- Strange attractors:
    - fractal "surfaces"
    - Resemble the Cantor set

References

- Lorenz, E.,“Computational Chaos”,Physica D, 1988