Proof that the lognormal PDF is heavier tailed than the gamma PDF.
Let $f(x)$ be the gamma PDF and $g(x)$ be the lognormal PDF.

$$\begin{aligned}
\log(f(x)) &= \log\left(\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\right) \\
&= \log\left(\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}\right) \\
&= \log\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right) - (\alpha-1)\log(x) - \frac{x}{\beta} \\
\log(g(x)) &= \log\left(\frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}\right) \\
&= \log\left(\frac{1}{\sqrt{2\pi}\sigma x}\right) - \frac{(\log(x)-\mu)^2}{2\sigma^2} \\
&= \log(1) - \log(\sqrt{2\pi}\sigma x) - \frac{(\log(x)-\mu)^2}{2\sigma^2}
\end{aligned}$$

Subtracting $\log(g(x))$ from $\log(f(x))$:

$$\begin{aligned}
&\log\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right) - (\alpha-1)\log(x) - \frac{x}{\beta} \\
&- \left(\log(1) - \log(\sqrt{2\pi}\sigma x) - \frac{(\log(x)-\mu)^2}{2\sigma^2}\right) \\
&= \log\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right) - (\alpha-1)\log(x) - \frac{x}{\beta} \\
&+ \log(\sqrt{2\pi}\sigma) + \log(x) + \frac{(\log(x)-\mu)^2}{2\sigma^2}
\end{aligned}$$

Expanding the quadratic and combining constants:

$$-(\alpha-1)\log(x) - \frac{x}{\beta} + \log(x) + \frac{\log(x)^2 - 2\mu\log(x)}{2\sigma^2} + C$$

Because the negative linear term dominates the highest order positive quadratic logarithmic term:

$$\begin{aligned}
&\lim_{x\to\infty}(\log(f(x)) - \log(g(x))) = -\infty \\
&\therefore \lim_{x\to\infty}\left(\log\left(\frac{f(x)}{g(x)}\right)\right) = -\infty \\
&\therefore \lim_{x\to\infty}\left(\log\left(\frac{f(x)}{g(x)}\right)\right) = -\infty
\end{aligned}$$

$$\lim_{y \to 0} \log(y) = -\infty$$

$$\therefore \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

Which implies that $f(x) \in o(g(x))$ and therefore $f(x)$ is heavier tailed than $g(x)$. QED.